NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 1101

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

CALCULUS/ANALYSIS

NOV/DEC 2001

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY FOUR** questions from Section B.

SECTION A

A1. Define a rational number. Show that between any two rational numbers there is another rational number. [1,3]

A2. Solve the following inequalities

(a) $|x^2 - 9| > 2$

(b) $\frac{x+3}{3x+1} < \frac{x}{x+2}$

[4,4]

A3. Find the derivative of $f(x) = \sqrt{x^2 + 1}$ from first principles. [5]

A4. State Leibniz's rule, and hence find $h^{(4)}(x)$ if $h(x) = e^{2x} \cos 3x$. [1,4]

A5. Prove by induction that

$$\sum_{k=1}^{n} k^2 = \frac{n}{6}(n+1)(2n+1)$$

[4]

A6. If $(f \circ g)(x) = \sqrt{\sin(x^2)}$ and $(g \circ f)(x) = \sin x$, find $f(x)$ and $g(x)$ [3]

A7. Evaluate $lim_{x\to\infty}(1+\frac{1}{x})^x$ [4]

A8. Evaluate

(a) $\int x\sqrt{x^2+1}\,dx$ [3]

(b) $\int_0^\infty \frac{dx}{1+x^2}$ [4]

SECTION B

B9. (a) Define a real-valued function. [2]

(b) State when a function is one to one. [2]

(c) If $f(x) = x^2 - 10x + 16$, $2 < x < 8$

(i) Find $f(-1)$ and $f(3)$ [1,1]

(ii) State the range of $f(x)$ [1]

(iii) After suitably restricting the domain of $f(x)$, find $f^{-1}(x)$ and state its domain. [4]

(iv) If $g(x) = \sqrt{x^2 - 8x + 16}$ find $(g \circ f)(x)$ [4]

B10. (a) State Rolle's theorem. [2]

(b) State and prove the Mean Value Theorem. [2,5]

(c) Prove that if $p < q$ then

$$\frac{q-p}{1+q^2} < tan^{-1}q - tan^{-1}p < \frac{q-p}{1+p^2}$$

[3]

(d) Show that $\frac{\pi}{4} + \frac{3}{25} < tan^{-1}(\frac{3}{4}) < \frac{\pi}{4} + \frac{1}{6}$ [3]

B11. (a) State L'Hopital's rule for indeterminant limits and hence evaluate

(i) $lim_{x \to 0^+} \frac{ln\cos(3x)}{ln\cos(2x)}$

(ii) $lim_{x \to 0^+} (\cos x)^{\frac{1}{x^2}}$

[1,4,4]

(b) Prove that $lim_{x \to 0^+} \frac{2}{1+exp(-1/x)} = 2$ [6]

B12. (a) If $f(x)$ is continuous in $[a,b]$ and $m \le f(x) \le M$ where $m, M \in \mathbf{R}$, prove that

$$m(b-a) \le \int_a^b f(x)\, dx \le M(b-a)$$

(b) Interpret the result geometrically.

(c) Prove that

$$lim_{n \to \infty} \int_0^{2\pi} \frac{\sin n\pi}{x^2 + n^2} dx = 0$$

[6,4,5]

B13. (a) Evaluate the following integrals using the given substitution.

(i) $\int_1^2 \frac{dx}{(x^2-2x+4)^{3/2}}$; $x - 1 = \sqrt{3}tan(u)$

(ii) $\int \frac{dx}{5+3\cos x}$; $tan(\frac{x}{2}) = u$

[3,4]

(b) Prove that $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx = \frac{\pi}{4}$ [3]

(c) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ approximately using Simpson's 1/3 rule with $n = 5$ and estimate the absolute error. [5]

B14. (a) Find the tangent to the right-hand parabola branch $x = sec(t), y = tan(t), -\frac{\pi}{2} < t < \frac{\pi}{2}$ at the point $(\sqrt{2}, 1)$.

(b) Find the length of the astroid $x = \cos^3 t, y = \sin^3 t, 0 \le t \le 2\pi$.

(c) You have been asked to design a one litre beer can shaped like a right circular cylinder. What dimensions will use the least material? Support your answer.

[5,5,5]

END OF QUESTION PAPER

Appendix B

Indefinite Integrals

Note that "+ C" has been omitted from all the integrals below.

- *Elementary Forms*

$$
\begin{aligned}
\int x^n\,dx &= \frac{x^{n+1}}{n+1} \quad (n \neq -1)\\
\int \frac{1}{x}\,dx &= \ln|x|\\
\int e^{kx}\,dx &= \frac{1}{k}e^{kx}\\
\int a^x\,dx &= \frac{1}{\ln a}a^x\\
\int \ln|x|\,dx &= x\ln|x| - x\\
\int \sin kx\,dx &= -\frac{1}{k}\cos kx\\
\int \cos kx\,dx &= \frac{1}{k}\sin kx\\
\int \tan kx\,dx &= \frac{1}{k}\ln|\sec kx|\\
\int \cot kx\,dx &= \frac{1}{k}\ln|\sin kx|\\
\int \sec kx\,dx &= \frac{1}{k}\ln|\sec kx + \tan kx|\\
\int \operatorname{cosec} kx\,dx &= -\frac{1}{k}\ln|\operatorname{cosec} kx + \cot kx|\\
\int \sec^2 kx\,dx &= \frac{1}{k}\tan kx\\
\int \operatorname{cosec}^2 kx\,dx &= -\frac{1}{k}\cot kx\\
\int \frac{1}{x^2+a^2}\,dx &= \frac{1}{a}\tan^{-1}\frac{x}{a}\\
\int \frac{1}{x^2-a^2}\,dx &= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|\\
\int \frac{1}{\sqrt{a^2-x^2}}\,dx &= \sin^{-1}\frac{x}{a}
\end{aligned}
$$

$$\int x^2\sqrt{a^2-x^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2}+\frac{a^4}{8}\sin^{-1}\frac{x}{a}$$

$$\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}$$

$$\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x}-\sin^{-1}\frac{x}{a}$$

$$\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

- *Forms involving* $\sqrt{x \pm a}$

$$\int \frac{dx}{x\sqrt{x+a}} = \frac{1}{\sqrt{a}} \ln \left| \frac{\sqrt{x+a} - \sqrt{a}}{\sqrt{x+a} + \sqrt{a}} \right| \quad , \quad a > 0$$

$$\int \frac{dx}{x\sqrt{x-a}} = \frac{2}{\sqrt{a}} \tan^{-1} \frac{\sqrt{x-a}}{\sqrt{a}} \quad , \quad a > 0$$

$$\int \frac{\sqrt{x+a}}{x}\, dx = 2\sqrt{x+a} + \sqrt{a} \ln \left| \frac{\sqrt{x+a} - \sqrt{a}}{\sqrt{x+a} + \sqrt{a}} \right| \quad , \quad a > 0$$

$$\int \frac{\sqrt{x-a}}{x}\, dx = 2\sqrt{x-a} + 2\sqrt{a} \tan^{-1} \frac{\sqrt{x-a}}{\sqrt{a}} \quad , \quad a > 0$$

$$\int \frac{dx}{x^2\sqrt{x+a}} = -\frac{\sqrt{x+a}}{ax} - \frac{1}{2a^{3/2}} \ln \left| \frac{\sqrt{x+a} - \sqrt{a}}{\sqrt{x+a} + \sqrt{a}} \right| \quad , \quad a > 0$$

$$\int \frac{dx}{x^2\sqrt{x-a}} = \frac{\sqrt{x-a}}{ax} + \frac{1}{a^{3/2}} \tan^{-1} \frac{\sqrt{x-a}}{\sqrt{a}} \quad , \quad a > 0$$

- *Forms involving* $\sqrt{x^2 \pm a^2}$ *or* $\sqrt{a^2 - x^2}$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{x}{a} \right|$$

$$\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \cosh^{-1} \frac{x}{a}$$

$$\int \frac{\sqrt{x^2 - a^2}}{x}\, dx = \sqrt{x^2 - a^2} - a \sec^{-1} \frac{x}{a}$$

$$\int x^2\sqrt{x^2 - a^2}\, dx = \frac{x}{8}(2x^2 - a^2)\sqrt{x^2 - a^2} - \frac{a^4}{8} \cosh^{-1} \frac{x}{a}$$

$$\int \frac{\sqrt{x^2 - a^2}}{x^2}\, dx = -\frac{\sqrt{x^2 - a^2}}{x} + \cosh^{-1} \frac{x}{a}$$

$$\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a}$$

$$\int \frac{\sqrt{x^2 + a^2}}{x}\, dx = \sqrt{a^2 + x^2} - a \ln \left| \frac{a + \sqrt{x^2 + a^2}}{x} \right|$$

$$\int x^2\sqrt{x^2 + a^2}\, dx = \frac{x}{8}(2x^2 + a^2)\sqrt{x^2 + a^2} - \frac{a^4}{8} \sinh^{-1} \frac{x}{a}$$

$$\int \frac{x^2}{\sqrt{x^2 + a^2}}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} - \frac{a^2}{2} \sinh^{-1} \frac{x}{a}$$

$$\int \frac{\sqrt{x^2 + a^2}}{x^2}\, dx = -\frac{\sqrt{x^2 + a^2}}{x} + \sinh^{-1} \frac{x}{a}$$

$$\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{a^2 x}$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$

$$\int \frac{\sqrt{a^2 - x^2}}{x}\, dx = \sqrt{a^2 - x^2} - a \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|$$

$$\int \frac{x^2}{\sqrt{a^2 - x^2}}\, dx = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\frac{x}{a}$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\frac{x}{a}$$

$$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2}$$

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\ln(1-x^2)$$

$$\int \sec^{-1} x\,dx = x\sec^{-1} x - \cosh^{-1} x$$

$$\int \sinh kx = \frac{1}{k}\cosh kx$$

$$\int \cosh kx = \frac{1}{k}\sinh kx$$

$$\int \tanh x\,dx = \ln(\cosh x)$$

$$\int \coth x\,dx = \ln|\sinh x|$$

$$\int \operatorname{sech} x\,dx = \tan^{-1}|\sinh x|$$

$$\int \operatorname{cosech} x\,dx = \ln\left|\tanh\frac{x}{2}\right|$$

- *Further Trigonometric Forms*

$$\int \sec x\tan x\,dx = \sec x$$

$$\int \operatorname{cosec} x\cot x\,dx = -\operatorname{cosec} x$$

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx \quad , n \neq 1$$

$$\int \cot^n x\,dx = -\frac{1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x\,dx \quad , n \neq 1$$

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\,\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx \quad , n \neq 1$$

$$\int \operatorname{cosec}^n x\,dx = -\frac{1}{n-1}\operatorname{cosec}^{n-2} x\,\cot x + \frac{n-2}{n-1}\int \operatorname{cosec}^{n-2} x\,dx \quad , n \neq 1$$

$$\int \sin^n x\cos^m x\,dx = -\frac{\sin^{n-1} x\cos^{m+1} x}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} x\cos^m x\,dx \quad , m+n \neq 0$$

$$\int \frac{dx}{1\pm\sin kx} = \mp\frac{1}{k}\tan\left(\frac{\pi}{4}\mp\frac{kx}{2}\right)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$