NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 1101

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

CALCULUS I

JANUARY 2004

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A: Answer ALL questions in this section [40].

A1. (a) Show that $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$

(b) Given that $f(x) = 2x^2 - 5x + 6$, use δ,ϵ method to prove that $\lim_{x\to 2} f(x) = 4$

(c) Evaluate $\lim_{x\to\infty}\left(\frac{2x-3}{3x-7}\right)^4$. [3,4,3]

A2. Solve the following inequality $\frac{2}{x-1} < \frac{3}{2x+1}$. [5]

A3. Find the derivative of $f(x) = \sin x$ from first principles. [5]

A4. (a) Find the exact value of the modulus, r and the argument, θ of $(1+i)^{22}$.

(b) Express $\frac{(1+i)^{11}}{(1-i)^{11}}$ in the form $x + iy$. [3,2]

A5. Use Leibniz Rule to find $\frac{d^6y}{dx^6}$ when $y = x^3 \cos x$. [4]

A6. Evaluate the following integrals:

(a) $\int (x+2)\sin(x^2+4x-6)dx$

(b) $\int \frac{\cot(\ln x)}{x}dx$

(c) $\int \frac{\sinh x}{\sqrt{\cosh x}}dx$. [2,2,3]

A7. Given the function $f(x) = \begin{cases} \sqrt{1-x^2}, & 0 \le x < 1 \\ 2, & x = 1 \\ 1, & 1 < x < 2 \end{cases}$

Find each of the following limits, if it exists:

(a) $\lim_{x\to 1^+} f(x)$

(b) $\lim_{x\to 1^-} f(x)$

(c) $\lim_{x\to 1} f(x)$. [2,2,2]

A8. (a) Show that $\operatorname{sech}^{-1}x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \quad 0 < x \le 1.$

(b) Prove that $\cosh^2 x - \sinh^2 x = 1$. [5,3]

SECTION B: Answer THREE questions in this section [60].

B9. (a) State and prove the Mean Value Theorem. [7]

(b) Use the Mean Value Theorem to show that

$$5\frac{1}{12} < \sqrt{26} < 5\frac{1}{10}$$

[7]

(c) Suppose $f'(x) = 2$ and that $f(0) = 5$ use the Mean Value Theorem to show that $f(x) = 2x + 5$ at every value of x [6]

B10. (a) Given that $t = \tan\frac{x}{2}$, show that $\sin x = \frac{2t}{1+t^2}$.

(b) Hence or otherwise evaluate $\int \frac{dx}{2 + 2\cos x - \sin x}$.

(c) The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the $y - axis$ to generate a solid. Find the volume of the solid.

(d) If $C_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$, use integration by parts to show that C_n satisfies the reduction formula

$$C_n = \frac{n-1}{n} C_{n-2}, \qquad n \geq 2$$

[4,6,5,5]

B11. (a) Express 5θ in terms of $\cos\theta$ and hence solve the equation

$$16x^5 - 20x^3 + 5x - 1 = 0$$

(b) Apply De Moivre's theorem to evaluate the integral

$$\int_0^{\frac{\pi}{2}} e^{3x} \cos 5x dx$$

(c) Solve the equation $z^4 - 7z^3 + 11z^2 + z + 34 = 0$ given that $z = 4 - i$ is one solution.

[7,7,6]

B12. (a) Sketch the graph of $y = \frac{x-1}{x^2(x-2)}$. [6]

(b) Sketch the graph of $y = \left|\frac{x-1}{x^2(x-2)}\right|$. [3]

(c) Show that the function

$$f(x) = \begin{cases} \sin \pi x, & x \leq 1 \\ x^3 - 1, & x > 1 \end{cases}$$

is continuous but not differentiable at $x = 1$. [4]

(d) Find the equation of the normal to the curve $x^2 - xy + y^2 = 7$ at the point $(1, 2)$. [4]

(e) Given $f(x) = \frac{1}{x}$, find $f'(x)$ from first principles. [4]

END OF QUESTION PAPER