NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 1102

DEPARTMENT OF APPLIED MATHEMATICS

BSc Honours in Applied Mathematics: Part I

Linear Algebra.

DECEMBER 2005

Time : 3 hours

Answer any **five** questions. All questions carry **equal** marks

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Q1. (a) Find the value of α so that the vectors $\mathbf{u} = (2, 3\alpha, -4, 1, 5)$, $\mathbf{v} = (6, -1, 3, 7, 2\alpha)$ are orthogonal. [3]

(b) Show that the vector equation of the line, l_1, joining the points with position vectors $\mathbf{a} = 10\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 12\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ is

$$\mathbf{r} = 10\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}).$$

Hence show that this line intersects with the line, l_2,

$$\mathbf{r} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k} + \mu(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$$

and find the position vector of the point of intersection. [8]

(c) Prove the parallelogram law, that is, $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$.
Hint: use the fact that $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$. [5]

(d) If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, show that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

[5]

Q2. (a) Define the following terms

(i) rank of a matrix,

(ii) trace of a matrix,

(iii) reduced row-echelon form

(iv) an elementary matrix

(v) an inconsistent system.

[5]

(b) Solve the following system of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned}$$

using the

(i) Gauss-Jordan method,

(ii) Cramer's method.

[5]

[6]

(c) Prove that if $A = [a_{ij}]$ is of order $m \times n$, $B = [b_{jk}]$ is of order $n \times p$ and $C = [c_{kl}]$ is of order $p \times q$, then

$$A(BC) = (AB)C.$$

[4]

Q3. (a) Find the rank of the matrix

$$A = \begin{pmatrix} 2 & 1 & 11 & 2 \\ 1 & 0 & 4 & -1 \\ 11 & 4 & 56 & 5 \\ 2 & -1 & 5 & -6 \end{pmatrix}.$$

[5]

(b) Given that A and B are $n \times n$ nonsingular square matrices, prove that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

[5]

(c) For a system of

$$\begin{aligned} x + 2y + -3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2, \end{aligned}$$

find the value of a so that the system has

(i) no solution,

[3]

(ii) a unique solution, [3]

(iii) infinitely many solutions. [4]

Q4. (a) Show that the projection of vector **a** in the direction of vector **b** is given by

$$(\mathbf{a}.\hat{\mathbf{b}})\hat{\mathbf{b}}$$

where $\hat{\mathbf{b}}$ is the unit vector. Hence, find the projection of the vector $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ in the direction of the vector $2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$. [5]

(b) (i) Prove that the volume V of the tetrahedron with adjacent edges $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ is

$$V = \left|\frac{1}{6}\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\right|.$$

[4]

(ii) Given that $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}, \mathbf{b} = \mathbf{i} - 3\mathbf{j} - 2\mathbf{k}, \mathbf{c} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, find the volume of the tetrahedron formed by the vectors. [3]

(c) The planes

$$x + 2y + z = 5 \quad \text{and} \quad 3x - y + 2z = 1$$

intersect on the line L.

(i) Find the parametric equation of L. [3]

(ii) Find the vector equation of the plane through the point $(2, 5, 1)$ which passes through L. [3]

(iii) Find the perpendicular distance of the plane in c(ii) from the origin. [2]

Q5. (a) Let **V** be the vector of all 2×2 matrices over a field F. Show whether or not W is a subspace of V where

(i) W consists of all 2×2 diagonal matrices. [3]

(ii) W consists of all 2×2 matrices with zero determinant. [4]

(b) Find a homogeneous system whose solution set W is generated by

$$\{(1, -2, 0, 3), \quad (1, -1, -1, 4), \quad (1, 0, -2, 5)\}.$$

[7]

(c) Determine whether the vectors:

$$(1, 1, 2), \quad (1, 0, 1), \quad (2, 1, 3)$$

span R^3. [6]

Q6. (a) Given that

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix},$$

(i) use the method of cofactors to find the inverse of A. [6]

(ii) hence or otherwise solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 5 \\ 2x + 5y + 3z &= 3 \\ x + 8z &= 17. \end{aligned}$$

[4]

(b) Write the matrix

$$B = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix},$$

as a product of elementary matrices. [4]

(c) Suppose that A is invertible and say it is row reducible to the identity matrix I by the sequence of elementary operations $e_1, ..., e_n$. Show that the sequence of elementary row operations to I yields A^{-1}. [6]

Q7. (a) Prove that if $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_r}$ are vectors in a space V, then the set W of all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_r}$ is a subspace of V. [4]

(b) Find a basis for the space spanned by the sequence of vectors:

$$v_1 = (1, -2, 0, 0, -3), \quad v_2 = (2, -5, -3, -2, 6),$$

$$v_3 = (0, 5, 15, 10, 0), \quad v_4 = (2, 6, 18, 8, 6).$$

[5]

(c) Let $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_r}\}$ be a set of vectors in R^n. If $r > n$, prove that S is a linearly dependent set. [5]

(d) Show that the following vectors in R^3 are linearly dependent:

$$(1, -2, 1), \quad (2, 1, -1), \quad (7, -4, 1).$$

Hence find the dimension of the space they generate. [6]

END OF QUESTION PAPER