NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 1103

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA 1103: DISCRETE MATHEMATICS

AUGUST 2004 SUPPLEMENTARY

Time : 3 hours

Candidates should attempt ALL questions from Section A and **ANY THREE** questions from Section B.

SECTION A: [25 marks]

Answer **ALL** questions from this section.

A1. Let A be a set and let $\{B_i\}_{i\in\mathbb{N}}$ be a family of sets. Show that
$A - \bigcap_{i\in\mathbb{N}} B_i = \bigcup_{i\in\mathbb{N}}(A - B_i)$ [6]

A2. (a) Define an order relation. [1]

(b) On the set $\mathbb{R}$ of real numbers, define a relation $\sim$ by $x \sim y \Leftrightarrow x^2 = y^2$. Determine whether or not $\sim$ is an equivalence relation. [6]

A3. (a) Prove the following identities on the non-empty sets A, B and C

(i) $(A - B) - C = A - (B \cup C)$ [3]

(ii) $(A - B) \cap C = (A \cap C) - (B \cap C)$ [3]

(b) Prove by induction the following Bernoulli's inequality:
$(1+\alpha)^n \geq 1 + n\alpha$ for all $n \in \mathbb{N}$ and $\alpha > -1$ [6]

SECTION B: [75 marks]

Answer **ANY THREE** questions from this section. Each question carries 25 marks.

B4. (a) Let $\{E_i\}_{i\in\mathbb{N}}$ be any family of non-empty sets. Define another family $\{F_j\}_{j\in\mathbb{N}}$ of sets by

$$F_n = E_n - \left(\bigcup_{i=1}^{n-1} E_i\right), \quad \forall n \in \mathbb{N}.$$ Prove that

(i) $F_i \cap F_j = \emptyset, \quad \forall i \neq j$ [5]

(ii) $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$ [8]

(b) Prove that if p is a prime number and $n \in \mathbb{N}$, then $n^p - n$ is divisible by p. [6]

(c) Prove that $\sqrt{2}$ is irrational. [6]

B5. (a) Let $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}$ be Borel sets. Define $X - Y$ by

$X - Y = \{z : z = x - y; x \in X; y \in Y\}$.

Prove that $\sup(X - Y) = \sup X - \inf Y$. [7]

(b) (i) Prove that if x and y are nonzero integers, then x and y have a unique greatest common divisor. [6]

(ii) Let p be a prime number and let x and y be integers. Prove that if p divides xy, then either p divides x or p divides y. [4]

(c) Suppose that $f : A \to B$ and $g : B \to C$ are arbitrary functions. Show that

(i) If gof is injective, then f is injective. [4]

(ii) If gof is surjective, then g is surjective. [4]

B6. (a) Let $f : A \to B$ and $g : B \to C$ be bijections. Prove that $(gof)^{-1} = f^{-1}og^{-1}$ [5]

(b) Define the following logical terms

(i) contradiction [1]

(ii) tautology [2]

(c) (i) The Lotto game in Zimbabwe requires one to guess correctly, in any order, six numbers out of a possible 45 numbers in order to win the first price. Find the least number of tickets one has to buy in order to be 100% certain of a first price given that each ticket allows one to put four different guesses? [5]

(ii) How many different permutations can one form from the Russian word zadacha? [4]

(iii) Find n if $C_n^{n+3} - C_{n-1}^{n+2} = 15(n+1)$ [8]

B7. (a) Which of the propositions P and Q should be true and which should be false if the proposition
$(\sim(\sim P \vee Q) \wedge Q) \Rightarrow Q$ is true? [10]

(b) Vitalis, Peter, Lazurus and Sunday wrote a Mathematicas Olympiad test. Three different journalists asked the students about the positions , from the first to the fourth, each had occupied and the following were the responses in all the three cases:
1) Sunday———-"I was first"; Peter———-"I was second"
2) Sunday———-"I was second"; Vitalis————"I was third"
3) Lazarus———"I was second"; Vitalis———-I was fourth"
How did they occupy the first four positions if we know that in each case only one statement is correct? [6]

(c) Prove that between any two rational numbers x and y such that $x < y$, there is an irrational number c such that $x < c < y$. [9]

B8. (a) Show that the quadratic function $f(x) = ax^2 + bx + c$ is

(i) strictly decreasing in $(-\infty, -\frac{b}{2a}]$ and strictly increasing in $[-\frac{b}{2a}, +\infty)$ if $a > 0$ [5]

(ii) strictly increasing in $(-\infty, -\frac{b}{2a}]$ and strictly decreasing in $[-\frac{b}{2a}, +\infty)$ if $a < 0$. [5]

(b) Prove that the function $f(x) = \frac{x}{x^2+1}$; $x \in \mathbb{R}$ is bounded. [7]

(c) Prove by induction that $11^{n+1} + 12^{2n-1}$ is a multiple of 133 for $n \in \mathbb{N}$ and $n \geq 1$. [8]

END OF QUESTION PAPER