NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 1103

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA 1103: DISCRETE MATHEMATICS SUPPLEMENTARY EXAMINATIONS

JULY 2005

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY FOUR** questions from Section B.

SECTION A: [40 marks]

Answer **ALL** questions from this section.

A1. (a) Prove that for any sets A, B

(i) $A/B = A \cap B'$

(ii) $A \times B = B \times A$ iff A = B

(b) If $A_1 \in P(A)$ and $B_1 \in P(B)$, prove that $A_1 \times B_1 \in P(A \times B)$

[3,6,4]

A2. (a) Let X be a set. When is $R \subseteq X \times X$ said to be an equivalence relation?

(b) Determine whether or not the following relations from $\mathbb{Z}$ to $\mathbb{Z}$ are equivalence relations:-

(i) $m\mathcal{R}n$ iff $m + n = 0$

(ii) mQn iff $m - n$ is even

[3,4,4]

A3. Give an example of a function $f : A \to B$ which is surjective but not injective. [4]

A4. Let $*$ be a binary operation on $\mathbb{R}$ defined by $\forall x, y \in \mathbb{R} : x * y = x^2 - y^2$

(a) Discuss whether $*$ is commutative or associative

(b) Does $\mathbb{R}$ have an identity with respect to $*$?

[6]

A5. Show that if $\sqrt{6}$ can be written as $\sqrt{6} = \frac{a}{b}$, $a, b \in \mathbb{N}$ where a and b have no common factor greater than 1, then a must be an even number.

Hence prove by contradiction, that $\sqrt{6}$ is an irrational number. [6]

SECTION B: [60 marks]

Answer **ANY FOUR** questions from this section. Each question carries 15 marks.

B6. (a) (i) Consider the function $f : x \to x^2 + x + 1$, $\mathbb{R} \to \mathbb{R}$.
State with reasons whether or not f is injective (i.e one to one) and/or surjective (i.e onto).

(ii) Describe a function g formed by restricting the domain and codomain of f that is bijective and write down the inverse of g.

(b) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \geq 0 \\ x + 1 & \text{if } x < 0 \end{cases}$$

Show that f is a bijection and find f^{-1}

[9,6]

B7. Use the principle of mathematical induction to prove that

(a) $1^2 + 2^2 + 3^2 + \ldots n^2 = \frac{1}{6}n(n+1)(2n+1)^2$

(b) $\alpha + \alpha e + \alpha e^2 + \ldots + \alpha e^n = \frac{\alpha(e^{n+1}-1)}{e-1}$ where α, e are real numbers with $e \neq 1$

(c) $1 + nx < (1+x)^n$ for $n \geq 2$, $x \in \mathbb{R}$, $x > -1$ and $x \neq 0$

[4,5,6]

B8. (a) Define $\mathcal{R}$ on $\mathbb{R}$ by $x\mathcal{R}y$ iff $x^2 = y^2$

(i) Prove that $\mathcal{R}$ is an equivalence relation on $\mathbb{R}$.

(ii) What are $[0], [2], [49]$?

(b) Define $\mathcal{R}$ on $\mathbb{R} \times \mathbb{R}$ by $(x, y)\mathcal{R}(u, v)$ iff $x^2 + y^2 = u^2 + v^2$.

(i) Prove that $\mathcal{R}$ is an equivalence relation on $\mathbb{R} \times \mathbb{R}$

(ii) What are $[(0,0)], [(1,1)], [(3,4)]$?

(iii) Describe $[(a,b)]$ for any fixed values of a, b both in set theoretical terms and geometrically.

[6,9]

B9. (a) Let A and B be statements and T a tautology. Give the truth table for each of the following compound statements:

(i) $\neg B \Rightarrow (A \vee \neg T)$

(ii) $T \rightarrow (A \wedge B)$

(b) Give the truth table for the following formula

$((p \wedge \neg q) \vee r) \rightarrow ((r \vee p) \longleftrightarrow (r \vee q))$

[4,4,7]

B10. (a) Establish the following tautologies

(i) $A \wedge A \longleftrightarrow A$

(ii) $\neg(A \wedge B) \longleftrightarrow (\neg A) \vee (\neg B)$

(iii) $\neg(A \vee B) \longleftrightarrow (\neg A) \vee (\neg B)$

(b) Define A/B to mean $\neg(A \wedge B)$ (this is the Sheffer stroke function). Express $\neg A$, $A \vee B$, $A \wedge B$, $A \rightarrow B$ and $A \longleftrightarrow B$ in terms of $/$.

(c) $P(A, B, C)$ is defined to be true if precisely one of A, B, C is true. Express P in terms of $\wedge, \vee, \neg$ and hence in terms of $/$.

[15]

B11. (a) Prove that for any sets A, B if $A \subseteq B$ then $P(A) \subseteq P(B)$.

(b) Let A, B, X, Y be sets such that $B = X \cup Y$.

Show that $A \times B = (A \times X) \cup (A \times Y)$

(c) Let X be a nonempty set and $P(X)$ its power set. Define a relation $\sim$ on $P(X)$ by

$A \sim B \longleftrightarrow A \subseteq B$

Investigate whether $\sim$ is reflexive, symmetric, anti-symmetric or transitive.

[15]

END OF QUESTION PAPER