NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 1103

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

DISCRETE MATHEMATICS

DECEMBER 2005

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A: Answer ALL questions in this section [40 marks].

A1. Let X be a non-empty set and $P(X)$ its power set. Define a relation $\sim$ on P by $A \sim B$ if and only if $A \subseteq B$. Investigate whether $\sim$ is reflexive, symmetric or transitive. [5]

A2. Using the definition of even integer and odd integer, give a direct proof that this statement is true for all integers n: if n is odd, then $5n + 3$ is even. [3]

A3. Prove that for any sets A, B, C we have $(A \cup B) \times C = (A \times C) \cup (B \times C)$. [10]

A4. (a) A compound proposition that is always true is a tautology, irregardless of what truth values its atomic proposition have. Show that the compound proposition $\neg P \wedge (P \vee Q) \Rightarrow Q$ is a tautology. [4]

(b) Express the statement "Everyone has exactly one best friend" in predicate logic. [5]

A5. Prove that a binary operation has at most one identity. [4]

A6. Let $\oplus$ be a binary operation on $\mathbb{Z}$ defined by $\forall m, n \in \mathbb{Z} : m \oplus n = m + nm$.

(a) Discuss whether $\oplus$ is

(i) commutative, [3]

(ii) associative. [3]

(b) Does $\mathbb{Z}$ have an identity with respect to $\oplus$? [3]

SECTION B: Answer THREE questions in this section [60 marks].

B7. (a) Let $P(x, y)$ denote the statement "x+y=y+x". What is the truth value of the proposition $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R}), P(x, y)$? [2]

(b) Let $Q(x, y)$ denote the statement "x+y=0". What is the truth value of the proposition $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R}), Q(x, y)$? [3]

(c) Express in predicate logic the statement "If somebody is female and is a parent, then this person is someone's mother." [5]

(d) Let P, Q, R be statements. Give truth tables for compound statements:

(i) $P \wedge \neg(Q \Rightarrow \neg R)$, [4]

(ii) $\neg\neg(\neg P \vee \neg\neg Q) \vee R$, [3]

(iii) $((P \wedge \neg Q) \vee R) \Rightarrow ((R \vee P) \Leftrightarrow (R \vee Q))$. [3]

B8. (a) Consider the following sets:

$$A = \{n \in \mathbb{N} : n \text{ is divisible by } 2\}$$
$$B = \{m \in \mathbb{N} : m \text{ is the sum of two odd numbers}\}$$

Show that $A = B$. [6]

(b) Consider the function $f : x \to x^2 + 1$, $\mathbb{R} \to \mathbb{R}$.

(i) State with reasons, whether or not f is bijective,

(ii) Describe a function g formed by restricting the domain and codomain of f that is bijective, and write down the inverse of g.

[9]

(c) Consider the relation R defined on the set $\mathbb{R}$ by $(m, n)R(p, q)$ if and only if

$$mq = np,$$

Show that R is an equivalence relation. [5]

B9. (a) If A and B are non-empty sets prove that

$$A \times B = B \times A \Leftrightarrow A = B.$$

[4]

(b) If $A_1 \in P(A)$ and $B_1 \in P(B)$ prove that

$$A_1 \times B_1 \in P(A \times B).$$

[4]

(c) Consider the function $f : x \to x^2 + x + 1, \mathbb{R} \to \mathbb{R}$.

(i) State with reasons whether or not f is injective and or surjective, [6]

(ii) Describe a function h formed by restricting the domain and codomain of f that is bijective. [6]

B10. (a) Use the Principle of Mathematical Induction to prove that

(i) $2^n < n!$ for all nonnegative integers $n \geq 4$, [6]

(ii) $2^{2n} - 1$ is divisible by 3 for all integers $n \geq 1$. [6]

(b) Consider the relation R defined on the set $\mathbb{R}\backslash\{0\}$ by xRy if and only if $xy > 0$.

(i) Show that R is an equivalence relation, [6]

(ii) Write down the equivalence classes: [1], [-1], [2]. [2]

B11. (a) How many people must be selected from a collection of 15 married couples to ensure that at least two of the persons chosen are married to each other? [3]

(b) The Lotto game in Zimbabwe requires one to guess correctly, in any order, six numbers out of a possible 45 numbers in order to win the first prize. Find the least number of tickets one has to buy in order to be 100% certain of a first prize given that each ticket allows one to put four different guesses? [5]

(c) How many different permutations can one form from the word "enemies" if all the letters of the word are used each time? [4]

(d) Find n if $C_n^{n+3} - C_{n-1}^{n+2} = 15(n+1)$ [8]

END OF QUESTION PAPER