NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 1111

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA 111: MATHEMATICS FOR SCIENCE SUPPLEMENTARY

JUNE/JULY 2002

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A: [25 marks]

Answer **ALL** questions from this section.

A1. (a) When a polynomial is divided by (x - 2), the remainder is 2, and when divided by (x - 3) the remainder is 5. When it is divided by (x - 2)(x - 3) the remainder is ax + b. Find values of a and b. [3]

(b) Express $\frac{1-7x}{(1+x)(1-3x)}$ in partial fractions and obtain the coefficients of x^r in its expansion in ascending powers of x. [5]

If this coefficient is denoted by a_r, show that, if s is any integer greater than zero, then $\frac{a_{s+2}-a_s}{a_{s+1}-a_{s-1}} = 3$ [3]

A2. (a) Find the value of x, given that $A^2 = A^{-1}$, where

$$A = \begin{pmatrix} 1 & x & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

[4]

(b) Find the minimum value of the sum of a positive number and its reciprocal. [5]

A3. Use De Moivre's theorem to prove that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

[5]

SECTION B: [75 marks]

Answer **ANY THREE** questions from this section. Each question carries 25 marks.

B4. (a) Let the matrix A_t be defined for all real numbers t by $A_t = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & t \\ 4 & 7-t & -6 \end{pmatrix}$

(i) For what values of t does A_t have an inverse? [3]

(ii) When $t = -3$, find all the vectors $\vec{x}$ that satisfy the vector equation $A_{-3}\vec{x} = \begin{pmatrix} 11 \\ 3 \\ 6 \end{pmatrix}$. [3]

(iii) When $t = 2$, determine a vector $\vec{z} \neq 0$ that is orthogonal to each vector of the form $A_2\vec{x}$ where $\vec{x}$ is an arbitrary vector in $\mathbb{R}^3$. [3]

(b) Prove that $f(x) = x^3$ is continuous for all $x = x_0$. [3]

(c) Form a quadratic equation with roots which exceed by 2 the roots of the quadratic equation $3x^2$ - (p-4)x - (2p+1)=0. [4]

Find the values of p for which the given equation has equal roots. [3]

(d) The equation $M^2 = aM + bI$, where a and b are scalars is satisfied by the matrix M given by

$$M = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

Find the values of a and b and use the equation to find the inverse of the matrix M. [6]

B5. (a) Solve the following systems of equations

(i)

$$\begin{aligned} 3x - y + 2z &= 5 \\ 2x + y - z &= 2 \\ 4x - 2y - 2z &= -1 \end{aligned}$$

[4]

(ii)

$$\begin{aligned} 2x - 4y + 9z &= 28 \\ 7x + 9y - 9z &= 5 \\ 7x + 3y - 6z &= -1 \end{aligned}$$

[4]

(b) If A(x) denotes the matrix

$$\begin{pmatrix} 2-x & 2x-2 \\ 1-x & 2x-1 \end{pmatrix}$$

prove that A(x) is singular if and only if x = 0. [4]

Prove also that A(x)A(y) = A(xy) and hence show that the square of A(-1) is the identity matrix. Find the inverse of A(2). [6]

(c) Prove the identities

(i) $\dfrac{\sin 2A}{1+\cos 2A} = \tan A \qquad (A \neq \frac{1}{2}(2n+1)\pi)$ [2]

(ii) $\dfrac{\sin 2A}{1-\cos 2A} = \cot A \qquad (A \neq n\pi)$ [2]

(iii) $\dfrac{1+\cos x}{1-\cos x} = \cot^2 \dfrac{1}{2}x \qquad (x \neq 2n\pi)$ [3]

B6. (a) Consider a tree that is planted at time $t = 0$ and let $P(t)$ be its current market value at time t. The present discounted value of the tree is $f(t) = e^{-rt}P(t)$ where r is the interest rate. Given that $r = 5\%$ and $P(t) = t^2 + 10t + 25$, when should this tree be cut in order to maximize its present discounted value? [4]

(b) (i) Let D be an $n \times n$ matrix such that $D^2 = 2D + 3I_n$. Prove that $D^3 = aD + dI_n$ for suitable values of a and b.(I_n denotes $n \times n$ identity matrix) [3]

(ii) Suppose that X is an $m \times n$ matrix and that $X^T X$ is not singular. Show that the matrix $A = I_m - X(X^T X)^{-1}X^T$ is idempotent, that is , $A^2 = A$. [4]

(iii) Let f and g be two continuous functions at $x = x_0$. Show that $f + g$ is also continuous at $x = x_0$. [3]

(c) What happens to the root(s) of the quadratic equation $ax^2 + bx + c = 0$ if $a \to 0$? $(b \neq 0)$ [5]

(d) If $z = \cos\theta + i\sin\theta$, show that

$$\begin{aligned} z + \frac{1}{z} &= 2\cos\theta \text{ and} \\ z^n + \frac{1}{z^n} &= 2\cos n\theta, n \in \mathbb{N} \end{aligned}$$

[6]

B7. (a) The real numbers r and θ, where r > 0 and $-\pi < \theta < \pi$ are such that
$r\cos\theta + 2r^2\cos 2\theta + 3r^3\cos 3\theta = 0$
$r\sin\theta + 2r^2\sin 2\theta + 3r^3\sin 3\theta = 0$
By writing $z = r(\cos\theta + i\sin\theta)$, show that $z = \frac{1}{3}(-1 \pm i\sqrt{2})$. [3]
Deduce the value of r and the two possible values of $\tan\theta$ [4]

(b) If the equation $x^2 - qx + r = 0$ has roots $x = (\alpha+2)$ or $x = (\beta+1)$, where α and β are the real roots of the equation $2x^2 - bx + c = 0$ and $\alpha \geq \beta$, find q and r in terms of b and/or c [6]
In the case $\alpha = \beta$, show that $q^2 = 4r + 1$ [3]

(c) The equation of a curve is

$$y = \frac{1 - x^2}{1 + x^2}$$

(i) Show that $\frac{dy}{dx} = -\frac{4x}{(1+x^2)^2}$ and obtain $\frac{d^2y}{dx^2}$ in terms of x [3]

(ii) The curve crosses the X-axis at A and B and the Y axis at C. Show that the tangents to the curve at A and B each pass through C . [6]

B8. (a) Calculate the following integrals

(i) $\int \frac{dx}{\sin x \cos x}$ [3]

(ii) $\int \frac{e^x}{\sqrt{e^{2x} - 2}} dx$ [3]

(iii) $\int \frac{x}{2^x} dx$ [3]

(iv) $\int_e^{e^2} \frac{dx}{x\ln x}$ [3]

(v) $\int_0^1 \frac{dx}{x^2 + 4x + 5}$ [3]

(b) Calculate the area of the figure bounded by the function $y = 4x - x^2$ and the X-axis. [3]

(c) Find the volume of the solid generated by rotating the circle $x^2 + (y - b)^2 = a^2$ about the X-axis given that $a \leq b$. [4]

(d) Calculate the equation of the tangent and normal to the curve $y = x^3 + 2x^2 - 4x - 3$ at the point $(-2; 5)$. [3]

END OF QUESTION PAPER