National University of Science and Technology

Department of Applied Mathematics

March 2003 Supplementary Examination

SMA 1201 Calculus of Several Variables

Time: **3 Hours**

Answer **all** questions of Part A and any **four** of Part B:

Part A: Answer **all** questions: Marks: **40**

1 Find the first and second order differentials of
$u = f(\sqrt{x^2 + y^2})$. [2:4]

2. Find the directional derivative of the function

$$z = 1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$$

at the point $M\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ in the direction of the inward normal to the ellipse
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [6]

3. Investigate the continuity of the function

$$f(x,y) = \frac{x}{3x + 5y}$$

at the point $(0,0)$. [5]

4. Prove that if $u = \sqrt{x^2 + y^2 + z^2}$, then $d^2u \geq 0$. [6]

5. Find the divergence of the vector

$$\mathbf{a} = \frac{\varphi(r)}{r}\mathbf{r},$$

where $r = |\mathbf{r}|$ is the distance from the coordinate origin to the variable point $M(x, y, z)$. [6]

6. Find the unit normal $\mathbf{n}$ to the surface $x^2 + 2y^2 - z^2 - 8 = 0$ at the point $(1,2,1)$. Deduce the equation of the tangent plane to the surface at this point. [6]

7. Evaluate the double integral

$$\iint_{(s)} xy\,dx\,dy,$$

over the region S which is bounded by the x-axis and an upper semicircle $(x-2)^2 + y^2 = 1$. [5]

Part B: **60** Marks: Answer any **four** questions. All the questions carry **equal** marks:

8a) The existence of partial derivatives of a function at a point does not imply, in general, the continuity of the function at that point. Show that the partial derivatives of the function

$$u = \begin{cases} \dfrac{2xy}{x^2+y^2} & \textbf{for} \quad x^2 + y^2 \neq 0 \\ 0 & \textbf{for} \quad x = y = 0 \end{cases}$$

exist and that the function is continuous with respect to each of the variables x and y , but is not continuous at the point $(0,0)$ with respect to these variables. [9]

b) Show that the function

$$z = xy + x\varphi\left(\frac{y}{x}\right)$$

satisfies the equation

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy + z.$$ [6]

9 a) Find the circulation of the vector field

$$\mathbf{a} = -y^3\mathbf{i} + x^3\mathbf{j}$$

around the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ [5]

b) Use the Stokes theorem to find the circulation of the vector

$$\mathbf{a} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$$

Around the contour

$$\begin{cases} x^2 + y^2 = 4 \\ z = 0 \end{cases}$$ [5]

c) Apply the Stokes' formula to find the integral

$$\oint_C (y+z)dx + (z+x)dy + (x+y)dz,$$

where C is the circle

$x^2 + y^2 + z^2 = a^2, \quad x + y + z = 0.$ [5]

10 a) The temperature at any point (x, y) in the Oxy – plane is given by

$$U(x,y) = \frac{100xy}{x^2 + y^2}.$$

(i) Find the directional derivative at (2,1) in a direction making an angle of 60 degrees with the positive x-axis. [5]

(ii) In what direction from (2,1) would the derivative be a maximum? [3]

(iii) What is the value of this maximum. [2]

b) Use Taylor's formula to find the change Δu for the function $u(x,y) = x^2 y + xy^2 - 2xy$ when a point (x, y) moves from $(1,-1)$ to $(1+h, -1+k)$.[5]

11. a) Prove that the function

$$u = \int_{-\infty}^{\infty} \frac{xf(z)}{x^2 + (y-z)^2} dz$$

satisfies the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$ [6]

b) Apply differentiation with respect to a parameter to evaluate the integral $\int_0^{\infty} \dfrac{e^{-ax} - e^{-bx}}{x} dx (a > 0, \quad b > 0)$. [5]

c) Prove that

$$\int_0^a \frac{dy}{\sqrt{a^4 - y^4}} = \frac{\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2}{4a\sqrt{2\pi}}.$$ [4]

12 a) Passing to polar coordinates, find the area bounded by the lines

$x^2 + y^2 = 2x, \quad x^2 + y^2 = 4x, \quad y = x, \quad y = 0.$ [5]

b) Sketch the solid whose volume is expressed by the double integral $\int_0^a dx \int_0^{\sqrt{a^2 - x^2}} \sqrt{a^2 - x^2 - y^2}\, dy$. Reason geometrically to find the value of the integral. [6]

c) Find the volume of a tetrahedron bounded by $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the coordinate planes. [4]

13 a) A rectangular tank, open at the top, is to hold 12 cubic meters of water. What must be the dimensions of this tank so that the total surface area is a minimum? [7]

b) (i) Show that the maximum and minimum values of $f(x,y)=x^2+xy+y^2$ in the unit square $0 \le x \le 1,\quad 0 \le y \le 1$ are 3 and 0 respectively. [6]

(ii) Explain why the above result connot be obtained by using a necessary condition that $f(x,y)$ has a maximum or minimum if $\frac{\partial f}{\partial x}=0,\ \frac{\partial f}{\partial y}=0$. [2]