NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 1201

BSc Honours in Applied Mathematics, PART I

SMA 1201 CALCULUS OF SEVERAL VARIABLES

JUNE 2004

Time : 3 hours

Attempt ALL questions in Section A and Any Two Questions in Section B.

SECTION A: Answer ALL questions in this section [50].

1. Find

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + xy^2 + yx^3 + y^3}{x^3 + y^2}$$

[2]

2. (a) Show that the vector function

$$\vec{F} = (6xy^2 + z)\hat{i} + (6x^2y + 3y^2)\hat{j} + x\hat{k}$$

is conservative and find its potential. [4]

(b) Show that $\nabla^2 r^n = n(n+1)r^{n-2}$ [5]

(c) Evaluate $\nabla^2(\ln r)$ [5]

3. (a) Find and classify the stationary points of

$$z = f(x,y) = (x^2 + y^2)^2 + 2(x^2 - y^2)$$

[7]

(b) Find, if it exists, the minimum value of

$$z = f(x,y) = x^2 + xy + y^2 + x - 4y + 9$$

[3]

4. For the space curve given by

$$\vec{r} = 2\cos t\hat{i} + 2\sin t\hat{j} + 5t\hat{k}$$

find:

(a) the unit tangent vector **T**, [3]

(b) the principal normal **N**, [3]

(c) the binormal vector **B**, [3]

(d) the curvature κ and radius of curvature ρ, and [3]

(e) the torsion τ and radius of torsion σ. [3]

5. For each of the following sketch the region of integration and evaluate the given integrals:

(a)

$$\int\int_R \sqrt{a^2 - x^2}dxdy$$

where R is the region bounded by $x^2 + y^2 \leq a^2$, $x \geq 0$, $y \geq 0$. [3]

(b)

$$\int\int_R \sqrt{x^2 + y^2}dxdy$$

where R is the region of the xy-plane bounded by $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. [3]

(c)

$$\int_{y=0}^{3} \int_{y=1}^{\sqrt{4-y}} (x + y)dxdy.$$

[3]

SECTION B: Answer TWO questions in this section [50].

6. (a) Verify that:

(i)
$$\nabla \times (\nabla \phi) = 0$$
[3]

(ii)
$$\nabla \cdot (\nabla \times \vec{A}) = 0$$
[3]

(b) If $\vec{F} = \nabla \phi$ where ϕ is single-valued and has continous partial derivatives, show that the work done in moving a particle from one point $P_1 = (x_1, y_1, z_1)$ to another point $P_1 = (x_2, y_2, z_2)$ is independent of the path joining the two points. [5]

(c) If $\vec{F}$ is irrotational (that is, $\nabla \times \vec{F} = 0$) prove that $\vec{F}$ is conservative. [10]

(d) Consider the motion of a particle of mass m in the earth's gravitional field represented by
$$\vec{F}(x, y, z) = (0, 0, -g).$$
Find the potential function $\phi(x, y, z)$ such that $\phi(0, 0, 0) = 0$. [4]

7. (a) State Green's theorem in the plane vectorially. [3]

(b) Prove Green's theorem in the plane: If P, Q, $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continous functions in a region R in the xy-plane bounded by a closed curve C, then
$$\oint_C Pdx + Qdy = \int\int_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$$
[10]

(c) Verify Green's theorem in the plane for
$$\oint_C (x^2y + y)dx + y^2dy$$
where C is the boundary of the region between $y = x^2$ and $y = x$. [9]

(d) Using Green's theorem evaluate the integral
$$\oint_C (6x + 2y)dx + (5x - 3y)dy$$
where C is the ellipse $9x^2 + y^2 = 9$ transversed in the positive direction. [3]

8. (a) State the divergence theorem and write it in rectangular form. [4]

(b) Use the divergence theorem to evaluate $\int\int_S \vec{F} \cdot \hat{n} dS$ where

$$\vec{F} = (x^2 + xz)\hat{\imath} + 2y^2\hat{\jmath} - \frac{z}{2}\hat{k}$$

and S is the surface of the cube bounded by
$x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0$ and $z = 1$. [5]

(c) Verify the divergence theorem for

$$\vec{F} = 2x^2\hat{\imath} - 3y\hat{\jmath} + z^2\hat{k}$$

where S is the region bounded by $x^2 + y^2 = 9,\ z = 0,\ z = 2$. [10]

(d) Let S be the boundary surface of a region D in space and let $\hat{n}$ be its outer normal. Prove the formulas:

$$V = \int\int_S xdydz = \int\int_S ydzdx = \int\int_S zdxdy = \frac{1}{3}\int\int_S xdydz + ydzdx + zdxdy$$

where V is the volume of D. [6]

9. (a) Write down Stokes' theorem in vector form and in rectangular form. [4]

(b) Verify Green's theorem for

$$\vec{F} = (x + 2y)\hat{\imath} + 3z\hat{\jmath} + yz\hat{k}$$

where S is the surface of the cube
$x = 0,\ y = 0,\ z = 0,\ x = 1,\ y = 1,\ z = 1$ above the xy-plane. [11]

(c) Evaluate using Stokes' theorem the integral,

$$\oint_C (2xy^2 + \sin z)dx + 2x^2ydy + x\cos zdz$$

around the curve $x = \cos t,\ y = \sin t\ z = \sin t,\ 0 \geq t \geq 2\pi$. [4]

(d) Let $\vec{F}$ be a continuously differentiable vector field such that

$$\vec{F} = f(x,y)\hat{\imath} + g(x,y)\hat{\jmath}.$$

Show that Stokes' theorem applied to a planar surface in the xy-plane yields Green's theorem. [6]

END OF QUESTION PAPER