# NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 1201

FACULTY OF APPLIED SCIENCES

**DEPARTMENT OF APPLIED MATHEMATICS**

SMA1201: CALCULUS OF SEVERAL VARIABLES

MAY 2005

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Sections B.

**SECTION A: Answer ALL questions in this section [40].**

**A1.** Convert $\left(\frac{2}{3}, \frac{\pi}{2}, \frac{\pi}{6}\right)$ from spherical coordinates to cylindrical coordinates. [4]

**A2.** (a) Using the definition of a limit, prove that $\lim\limits_{(x,y)\to(1,2)} (3x + 2y) = 7$. [5]

(b) Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^y \sin x}{x}$ . [2]

**A3.** If $z = f(ax^2 + by^2)$, show that $(z_{xx} - x^{-1}z_x)(z_{yy} - y^{-1}z_y) = (z_{xy})^2$. [6]

**A4.** Let $z = f(x, y) = x^2y - 3y$. Evaluate $f(4, 3)$ and hence find $f(3.98, 3.02)$ using the method of differentials. [4]

**A5.** The Human Cardiovascular system is similar to electrical series and parallel circuits. For example, when blood flows through two resistances $R_1$ and $R_2$ in parallel, then the equivalent resistance $R$ of the network is

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \text{ ,or } R = \frac{R_1 R_2}{R_1 + R_2}$$

If the percentage errors in measuring $R_1$ and $R_2$ are $\pm 0.6\%$ and $\pm 0.9\%$ respectively, find the approximate maximum percentage error in $R$. [4]

**A6.** The temperature in a rectangular box is approximated by
$T(x, y, z) = xyz(1 - x)(2 - y)(3 - z), \quad 0 \le x \le 1, \ \ 0 \le y \le 2, \text{ and } \ 0 \le z \le 3.$
If a mosquito is located at $\left(\frac{1}{2}, 1, 1\right)$, in which direction should it fly in order to cool off as rapidly as possible? [5]

**A7.** By reversing the order of integration, evaluate

$$\int_0^1 \int_0^{\cos^{-1} y} x dx dy.$$

[5]

**A8.** For the vector field $\vec{F} = 4xyz\mathbf{i} + (2x^2z + z)\mathbf{j} + (2x^2y + 3 + y)\mathbf{k}$, determine if the field is conservative and , if it is , find the scalar potential. [5]

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**SECTION B: Answer THREE questions in this section [60].**

**B9.** (a) Express the Divergence theorem in words. [2]

(b) Prove the Divergence theorem. [9]

(c) Verify the Divergence theorem for $\vec{F} = 4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k}$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$. [9]

**B10.** (a) A revenue function is

$R(x, y) = x(100 - 6x) + y(192 - 4y),$

where $x$ and $y$ denote the number of items of two commodities sold.

Given that the corresponding cost function is

$C(x, y) = 2x^2 + 2y^2 + 4xy - 8x + 20,$

find the maximum profit. [Hint: profit= revenue − cost]. [8]

(b) Find the greatest and smallest values that the function

$$f(x, y) = xy$$

takes on the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$ [8]

(c) Sketch the surface

$$z = 4 - x^2 - y^2.$$ [4]

**B11.** (a) With reference to vector fields, explain what each of the following statements mean :

(i) irrotational. [1]

(ii) solenoidal. [1]

(iii) vortex. [1]

(b) Determine the constants $\alpha$ and $\beta$ such that the vector field $\vec{F} = (x + \alpha y)\mathbf{i} + (y + \beta x)\mathbf{j} + z\mathbf{k}$ is irrotational. [2]

(c) If $\vec{E}$ is a solenoidal vector field whose two components are $b_1 = x^2y$ and $b_2 = e^x y + y^2$, determine the third component, $b_3$. [2]

(d) Find $\text{curl}(\vec{r}f(r))$, where $f(r)$ is differentiable. [5]

(e) Prove that $\nabla r^n = nr^{n-2}\vec{r}$. [4]

(f) If $\vec{v} = \vec{\omega} \times \vec{r}$, prove that $\vec{\omega} = \frac{1}{2}$ curl $\vec{v}$, where $\vec{\omega}$ is a constant vector. [4]

**B12.** (a) Let $R$ be the region in the $xyz$-space defined by the inequalities $1 \leq x \leq 2,\ 0 \leq xy \leq 2$, and $0 \leq z \leq 1$.
Evaluate

$$\int\int_R\int \left(x^2y + 3xyz\right)\, dxdydz$$

by applying the transformation

$u = x$
$v = xy$
and $w = 3z$

and integrating over an appropriate region $G$ in $uvw$-space. [12]

(b) Evaluate $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\sec\phi}^{2} 3\rho^2 \sin\phi\, d\rho d\phi d\theta.$ [4]

(c) Prove that if $u = g(x, y), v = h(x, y), x = k(s, t)$ and $y = r(s, t)$ then:

$$\frac{\partial(u, v)}{\partial(s, t)} = \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(s, t)} .$$

[4]

**B13.** (a) Prove Green's theorem in the plane if $C$ is a closed curve which has the property that any straight line parallel to the coordinate axes cuts $C$ in at most two points. [8]

(b) Show that Green's theorem is a special case of Stokes' theorem. [6]

(c) Verify Green's theorem for the line integral

$$\oint_C y^3 dx + (x^3 + 3xy^2) dy,$$

where $C$ is the path from $(0, 0)$ to $(1, 1)$ along the graph $y = x^3$ and from $(1, 1)$ to $(0, 0)$ along the graph of $y = x$. [6]

**END OF QUESTION PAPER**