NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 1201

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA1201: CALCULUS OF SEVERAL VARIABLES

JULY 2005 SUPPLEMENTARY EXAMINATIONS

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Sections B.

SECTION A: Answer ALL questions in this section [40].

A1. Evaluate $\nabla^2(\ln r)$ where r has its usual meaning. [4]

A2. (a) Give the $\epsilon - \delta$ definition of a limit of a function of two variables. [4]

(b) Show that $\lim_{(x,y)\to(0,0)} \left(\frac{xy^2}{x^2+y^4}\right)$ does not exist. [4]

A3. If $z = f(ax^2 + by^2)$, show that $(z_{xx} - x^{-1}z_x)(z_{yy} - y^{-1}z_y) = (z_{xy})^2$. [6]

A4. (a) Using polar co-ordinates, evaluate

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy$$

[7]

(b) Deduce that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

[3]

A5. If $\vec{F} = \nabla\phi$, where ϕ is single-valued and has continuous partial derivatives, show that the work done in moving a particle from one point $P_1(x_1, y_1, z_1)$ in this force field to another point $P_2(x_2, y_2, z_2)$ is independent of the path joining the two points. [7]

A6. The temperature in a region of space is $cy^2(x - z)$, where c is a positive constant. If an insect sets off from the point $(1, 1, 2)$ with given speed v, in an attempt to get warm as fast as possible, in what direction should it fly? [5]

SECTION B: Answer THREE questions in this section [60].

B7. (a) Express the Divergence theorem in words. [2]

(b) Prove the Divergence theorem. [9]

(c) Verify the Divergence theorem for $\vec{F} = 4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k}$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$. [9]

B8. (a) Find and classify all the critical points of

$$z = x^3 + y^3 - 3x - 12y + 20.$$

Evaluate z at each critical point and state the maximum value z can take. [8]

(b) Find the greatest and smallest values that the function

$f(x, y) = xy$

takes on the ellipse

$\frac{x^2}{8} + \frac{y^2}{2} = 1.$ [8]

(c) Sketch the surface

$y^2 - x^2 = 16.$ [4]

B9. (a) Let $f(x, y)$ be a function of x and y, where $x = e^{u-v}\cos(u+v)$, $y = e^{u-v}\sin(u+v)$. Show that $\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} = 2x\frac{\partial f}{\partial y} - 2y\frac{\partial f}{\partial x}$. [10]

(b) Parabolic co-ordinates (u, v) are defined implicitly in terms of cartesian co-ordinates (x, y) by the pair of equations $x = \frac{u^2 - v^2}{2}$, $y = uv$. Given that $f(x, y) = \phi(u, v)$, obtain expressions for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in terms of $\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v}$, u and v and deduce that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = (u^2 + v^2)^{-1}\left[\left(\frac{\partial \phi}{\partial u}\right)^2 + \left(\frac{\partial \phi}{\partial v}\right)^2\right]$$

[10]

B10. (a) Evaluate the integral

$$\int_0^3 \int_0^4 \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \left(\frac{2x - y}{2} + \frac{z}{3}\right) dxdydz$$

by applying the transformation

$$u = \frac{2x - y}{2}, v = \frac{y}{2}, w = \frac{z}{3}$$

and integrating over an appropriate region in the uvw-space. [12]

(b) Evaluate $\int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_{\sec\phi}^{2} 3\rho^2 \sin\phi d\rho d\phi d\theta$. [4]

(c) Prove that if $u = g(x, y), v = h(x, y), x = k(s, t)$ and $y = r(s, t)$ then:

$$\frac{\partial(u, v)}{\partial(s, t)} = \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(s, t)}.$$

[4]

B11. (a) Prove Green's theorem in the plane if C is a closed curve which has the property that any straight line parallel to the coordinate axes cuts C in at most two points. [7]

(b) Verify Green's theorem in the plane for

$$\oint_C (3x - 2y)dx + (x - 4y)dy,$$

where C is the boundary of the simply-connected region R enclosed by the circles $x^2 + y^2 = 16$ and $x^2 + y^2 = 1$. [10]

(c) If $\mathbf{F} = (x^2 + y^2)\mathbf{i} - 3xz\mathbf{j} + y^3\mathbf{k}$, compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ from $(0, 0, 0)$ to $(1, 1, 1)$. [3]

END OF QUESTION PAPER