NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA 1202 REAL ANALYSIS

Nov/Dec 2002 **Time 3 Hours**

Candidates should attempt **ALL** questions from Section **A** and **ANY FOUR** questions from Section B.

SECTION A

A1 Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{2^n}{n+1} x^n$

[5]

A2. Let $a_n = \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$

Using the identify

$$\frac{2}{r(r+1)(r+2)} = \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$$

Verify that

$$a_n = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right)$$

By using the $\in - N$ terminology, prove that (a_n) converges to ¼. [7]

A3. Let f: $(0,1] \to \mathbf{R}$ be given by $f(x) = \frac{1}{x}$ for all $x \in (0,1]$. Show that f is continuous at each $x \in (0,1]$, but f is not uniformly continuous there. [7]

A4. Let X be a non empty set. Define what is meant by a <u>metric</u> d on X. For each of the following pairs (X, d) determine whether d is a metric on the set X.

(You should show either that all the axioms for a metric hold, or that one of the axioms fails).

(i) $X = \mathbf{R}$, $d(x, y) = |\sin x - \sin y|$ $(x, y \in \mathbf{R})$

(ii) $X = \mathbf{R}^2$, $d(x, y) = |x_1 - y_1| + 2|x_2 - y_2|$ $(x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbf{R}^2)$

[10]

A5. Let (X,d) be a metric space and let (x_n) be a sequence in X

(a) State what is meant by the statements

(i) (x_n) converges to $x \in X$, and

(ii) (x_n) is a Cauchy sequence.

(b) Prove that if (x_n) converges to $x \in X$, and (x_n) converges to $y \in X$ then $x = y$.

(c) Prove that every convergent sequence in (X, d) is a Cauchy sequence.

[11]

SECTION B

B6 (a) Let $\{x_n\}$ be a sequence of real numbers satisfying

(i) $x_1 = ½$

(ii) $x_{n+1} = \dfrac{1}{2 - x_n}$

Show that the sequence tends to a limit and evaluate the limit.

(b) Show that the sequence $\{a_n\}$ given by:

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log(n+1)$$

converges to a limit l, say.

Hence, or otherwise, evaluate the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + - - - -$$

[8,7]

B7 (a) Giving reasons for each step taken, prove that

$$ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - - - -, \; for \; -1 < x \leq 1$$

You may assume that

$$\frac{1}{1-x} = 1 + x + x^2 + - - - - + x^n + - - - -, \; for \; -1 < x < 1.$$

Deduce that

$$ln\,2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - - - -$$

(b) Write down the derivative of cos (cos x). Use the Mean Value Theorem to show that for all x and y.

$$|\cos(\cos x) - \cos(\cos y)| < \frac{1}{2}\sqrt{3}\,|x - y|.$$

[Hint: $\sin(1) < \sin\frac{\pi}{3}$]

[9,6]

B8. (a) When is a function f said to be continuous at a point x?

(b) Show that if f and g are both continuous at a point x_0, then

(i) the function $f + g$ is continuous at x_0.
(ii) the function fg is continuous at x_0.

(c) Let f be the function defined by

$$f(x) = \begin{cases} x \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

and let g be the function defined by $g(x) = xf(x)$. Show that f is continuous at $x = 0$. Show that g is differentiable at $x = 0$, and find its derivative there. Show that f is not differentiable at $x = 0$.

[2,6,7]

B9. (a) What is meant by saying that a sequence $\{a_n\}_{n=0}^{\infty}$ is

(i) monotonic increasing?

(ii) bounded above?

(b) What can be said about a sequence which is both monotonic increasing and bounded above?

(c) Let $0 < a_0 < b_0$ and define sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ recursively by

$$a_{n+1} = \frac{2a_n b_n}{a_n + b_n}, \quad b_{n+1} = \frac{1}{2}(a_n + b_n)$$

If $0 < a_n < b_n$ for some n, deduce that $0 < a_n < a_{n+1} < b_{n+1} < b_n$ and prove by induction that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are respectively monotonic increasing and monotonic decreasing.

(d) Deduce that both sequences converge to a common limit. [4,2,7,2]

B10 (a) Define what is meant by radius of convergence of a power series

$$\sum_{n=0}^{\infty} a_n x^n$$

(b) Prove that the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$ and that of the corresponding series of the derivatives $\sum_{n=1}^{\infty} na_{n-1} x^{n-1}$ are equal.

(c) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$

Find also the interval of convergence of the power series whose terms are derivatives of the terms of the given power series. Verify that the radii of convergence of the two power series are equal.

[2, 6, 7]

B11. Consider the series $\sum_{k=1}^{\infty} \frac{k}{(k+1)}$

(a) Find the partial sums s_1, s_2, s_3 and s_4. Use the pattern to guess a formula for s_n.

(b) Use induction to prove that your guess is correct.

(c) Show that the given infinite series is convergent and find its sum. **[2, 5, 8]**

END OF QUESTION PAPER