# NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 1202

## FACULTY OF APPLIED SCIENCES

### DEPARTMENT OF APPLIED MATHEMATICS

REAL ANALYSIS

MARCH 2003

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

### SECTION A: Answer ALL questions in this section [40].

**A1.** Investigate the behavior (by sketching the graph) of the following sequence.

$$S_n = \begin{cases} \frac{n}{n+1} & , \text{ when } n \text{ is odd} \\ 1 - \frac{1}{n} & , \text{ when } n \text{ is even} \end{cases}$$

If the sequence is convergent, state its limit $l$. [5]

**A2.** (a) When is a function $f$ said to be continuous at a point $x_0$? [2]

(b) Show that if $f$ and $g$ are both continuous functions at a point $x_0$, then

(i) the function $f + g$ is continuous at $x_0$,

(ii) the function $fg$ is continuous at $x_0$. [4,4]

**A3.** (a) What condition must a continuous function $f$ satisfy for its inverse $f^{-1}$ to exist? [2]

(b) Suppose $y = \cosh^{-1}(x)$. Obtain the derivative $\frac{dy}{dx}$. [3]

**A4.** Let $X$ be a nonempty set. Define what is meant by a metric $d$ on $X$. For each of the following pairs $(X, d)$ determine whether $d$ is a metric on the set $X$. (You should show either that all the axioms for a metric hold, or that one of the axions fails.)

(a) $X = \mathbb{R}, d(x, y) = |\sin x - \sin y|, (x, y \in \mathbb{R})$.

(b) $X =$ the set of triples $(x_1, x_2, x_3)$ where $x_i$ is 0 or 1, $d(x, y) =$ the number of values of $i = \{1, 2, 3\}$ such that $x_i \neq y_i$ $(x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in X)$. [10]

**A5.** Define a Cauchy sequence. Prove that the following sequence is Cauchy:

$$x = 1, \quad x_{n+1} = \frac{1}{2 + x_n}.$$

[10]

**SECTION B: Answer FOUR questions in this section [60].**

**B6.** (a) Let

$$f(x) = \begin{cases} \frac{|x-2|}{x-2} & , \text{ if } x \neq 2 \\ 0 & , \text{ if } x = 2 \end{cases}$$

(i) Give a sketch of the graph of $f(x)$.

(ii) Find $\lim\limits_{x \to 2^+} f(x)$.

(iii) Find $\lim\limits_{x \to 2^-} f(x)$.

(iv) Find $\lim\limits_{x \to 2} f(x)$.

(b) Consider a sequence $\{s_n\}$ given by

$$s_1 = 1, \text{ and } s_{n+1} = \frac{s_n}{1 + s_n}.$$

(i) Find the limit $l$ of $\{s_n\}$.

(ii) If $f(x) = 2 + x$, write out the first five terms of the (new) sequence and show $\{f(s_n)\}$ and show that $f(l)$ is the limit of this sequence. [15]

**B7.** Consider the sequence defined by

$$x_1 = 1, \quad x_2 = 2, \quad \text{and, for } n > 2, \; x_n = \frac{1}{2}[x_{n-1} + x_{n-2}].$$

(a) Show by induction that

$$|x_n - x_{n+1}| = \frac{1}{2^{n-1}}$$

and hence deduce that if $m > n$ then $|x_n - x_m| < \frac{1}{2^{n-1}}$. Hence show that the sequence is a Cauchy sequence.

(b) Given that the odd terms of the sequence are

$$x_{2n+1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots + \frac{1}{2^{2n-1}}$$

find the limit of the sequence. [15]

**B8.** (a) Let $\{x_n\}$ be a sequence of real numbers satisfying

(i) $x_1 = \frac{1}{2}$

(ii) $x_{n+1} = \frac{1}{2-x_n}$

Show that the sequence tends to a limit and evaluate the limit.

(b) Show that the sequence $\{a_n\}$ given by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log(n+1)$$

converges to a limit $l$, say.
Hence or otherwise, evaluate the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

[15]

**B9.** (a) Use the Binomial Theorem to show that for a fixed constant $b > 0$ and positive integer $n$:

$$(1+b)^n \geq 1 + nb.$$

(b) Suppose that $a > 1$. Writing

$$a^{\frac{1}{n}} = 1 + b_n \quad \text{(which serves to define } b_n\text{)},$$

use (a) to show that as $n \to \infty, b_n \to 0$. Deduce that $a^{\frac{1}{n}} \to 1$ [15]

**B10.** (a) State the Mean Value Theorem.
Using this or otherwise, show that if $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$ and $f'(x) = 0, \forall x \in (a, b)$ then $f$ is a constant on $[a, b]$

(b) Find $\lim_{x \to 0} \frac{1}{x}\left(x - \frac{1}{x}\right)$.

(c) Show that if $g(x)$ is given by

$$g(x) = \begin{cases} x\cos\frac{1}{x} & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0 \end{cases}$$

then $g(x)$ has no derivative at 0, but is continuous there. [15]

**B11.** (a) Prove that for $x > 0$ and any $k \in \mathbb{N}$

$$x - \frac{1}{2}x^2 + \cdots \frac{(-1)^{2k-1}}{2k}x^{2k} < \log(1+x) < x - \frac{1}{2}x^2 + \cdots \frac{x^{2k+1}}{2k+1}$$

(b) Evaluate

$$\lim_{x\to 0} \frac{(\cos bx)\ln(1+x^2) - x^2}{x^6}.$$

(c) Use Taylor's Theorem with $n = 2$ to approximate

$$\sqrt[3]{1+x}, \quad x > -1.$$

[15]

**END OF QUESTION PAPER**