

FACULTY OF APPLIED SCIENCES
DEPARTMENT OF APPLIED MATHEMATICS
SMA 1202: REAL ANALYSIS

JUNE 2004

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A: [25 marks]

Answer **ALL** questions from this section.

- A1.** Prove that \mathbb{R} is dense. [3]
- A2.** Prove that the set $X \subset \mathbb{R}$ is bounded if and only if $\exists a \in \mathbb{R} \forall x \in X$ such that $|x| \leq a$. [10]
- A3.** (a) Find the formula for the general term in the Fibonacci sequence:
 $x_1 = 1 ; x_2 = 1 ; x_n = x_{n-1} + x_{n-2} ; n \in \mathbb{N}; n \geq 3$ [6]
- (b) Define the limit of a sequence. [2]
- (c) Prove that $\lim_{x \rightarrow 0^+} \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$. [4]

SECTION B: [75 marks]

Answer **ANY THREE** questions from this section. Each question carries 25 marks.

- B4.** (a) Prove that the sequence $x_n = \frac{(-1)^n n + 10}{\sqrt{n^2 + 1}}$, ($n \in \mathbb{N}$); is bounded. [6]

- (b) Define a monotonic sequence. [3]
- (c) Show that the sequence $x_n = \left(1 + \frac{1}{n}\right)^n$ is strictly increasing. [7]
- (d) Show that every unbounded sequence contains a subsequence which is strictly increasing. [9]
- B5.** (a) Let $S(p)$ denote the sum of the digits of the natural number p . (If $p \leq 9$ then $S(p) = p$).
- (i) Let $x_1 \in \mathbb{N}$; $x_{n+1} = S(x_n)$ $n \in \mathbb{N}$.
Show that there exists some $N \in \mathbb{N}$ such that $\forall m, n \geq N$, then $x_n = x_m$. [6]
- (ii) Let $x_1 \in \mathbb{N}$; $x_{n+1} = S(S(x_n) + x_n)$; $n \in \mathbb{N}$
Calculate x_{1984} if $x_1 = 1983$. [5]
- (b) Prove that $\lim_{n \rightarrow \infty} x_n = a$ if and only if $\forall n \in \mathbb{N}$, $x_n = a + \alpha_n$ where $\{\alpha_n\}$ is a sequence which is infinitely small. [6]
- (c) Prove that $\lim_{n \rightarrow \infty} x_n = 1$, where $x_n = \frac{n}{n+1}$. [3]
- (d) Suppose that $\lim_{n \rightarrow \infty} x_n = 0$ and $x_n \geq -1 \forall n \in \mathbb{N}$.
If $p \in \mathbb{N}$, prove that $\lim_{n \rightarrow \infty} \sqrt[p]{1 + x_n} = 1$. [5]
- B6.** (a) Define the upper limit and lower limit of a sequence $\{X_n\}$. [4]
- (b) State the Bolzano-Weierstrass theorem. [2]
- (c) Use the Weierstrass theorem to prove that:
- (i) the sequence $\{x_n\}$, where $x_1 = 0$; $x_{n+1} = \sqrt{6 + x_n}$; ($n \in \mathbb{N}$); has a finite limit. [5]
Calculate this limit. [3]
- (ii) the sequence $\{x_n\}$ where $x_n = \left(1 + \frac{k}{n}\right)^n$ has a finite limit. [4]
Calculate this limit. [2]
- Hint:** Use the fact that $\lim_{p \rightarrow \infty} \left(1 + \frac{1}{p}\right)^p = e$.
- (d) Calculate the upper and lower limits of $x_n = \frac{n^2 \sin\left(\frac{n\pi}{2}\right) + 1}{n+1}$. [5]
- B7.** (a) State the Cauchy condition for convergence of a sequence. [2]
- (b) Prove that the sequence $x_n = \sum_{k=1}^n \frac{1}{k}$ ($n \in \mathbb{N}$) diverges. [5]
- (c) Calculate the following limits:
- (i) $\lim_{n \rightarrow \infty} \left(\frac{2^n + 1}{2^n}\right)^{2^n}$. [3]

$$(ii) \lim_{n \rightarrow \infty} \left(\frac{2002}{n} \right)^n \quad [3]$$

$$(iii) \lim_{n \rightarrow \infty} \sqrt{n^2 + n} - \sqrt{n^2 - n}. \quad [3]$$

$$(iv) \lim_{n \rightarrow \infty} \frac{10^n + n!}{2^n + (n+1)!}. \quad [3]$$

(d) Use the comparison test to determine whether $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ converges or diverges. [6]

B8. (a) State the following convergence tests for series:

(i) D'Alembert's criterion [2]

(ii) Cauchy's integral test [2]

(iii) Leibniz's convergence test. [2]

(b) Investigate the convergence of:

$$(i) \sum_{n=1}^{\infty} \frac{1}{n!} \quad [3]$$

$$(ii) \sum_{n=1}^{\infty} \left(\frac{3n}{3n+1} \right)^n \quad [3]$$

$$(iii) \sum_{n=1}^{\infty} \frac{3^n n!}{n^n} \quad [3]$$

$$(iv) \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2(n+2)^2} \quad [3]$$

$$(v) \sum_{n=1}^{\infty} \frac{3n-1}{(\sqrt{2})^n} \quad [4]$$

$$(vi) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}. \quad [3]$$

END OF QUESTION PAPER