NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 1202

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA 1202: REAL ANALYSIS SUPPLEMENTARY

AUGUST 2004

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A: [25 marks]

Answer **ALL** questions from this section.

A1. Let $\alpha, \beta \in \mathbb{R}$. Prove that if $\{x_n\}$ and $\{y_n\}$ are two sequences which are bounded, then $\{\alpha x_n + \beta y_n\}$ is also bounded. [5] [3]

A2. Prove that the set $X \subset \mathbb{R}$ is bounded if and only if $\exists a \in \mathbb{R}\ \forall x \in X$ such that $|x| \leq a$. [10]

A3. (a) Find the formula for the general term in the Fibonacci sequence:
$x_1 = 1$; $x_2 = 1$; $x_n = x_{n-1} + x_{n-2}$; $n \in \mathbb{N}$; $n \geq 3$ [6]

(b) Define the limit of a sequence. [2]

(c) Prove that $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1} = 2$. [4]

SECTION B: [75 marks]

Answer **ANY THREE** questions from this section. Each question carries 25 marks.

B4. (a) Prove that the sequence $x_n = \dfrac{(-1)^n n + 10}{\sqrt{n^2+1}}$, $(n \in \mathbb{N})$; is bounded. [6]

(b) Define a monotonic sequence. [3]

(c) Show that the sequence $x_n = \left(1 + \dfrac{1}{n}\right)^n$ is strictly increasing. [7]

(d) Show that every unbounded sequence contains a subsequence which is strictly increasing. [9]

B5. (a) Let $S(p)$ denote the sum of the digits of the natural number p. (If $p \leq 9$ then $S(p) = p$).

(i) Let $x_1 \in \mathbb{N}$; $x_{n+1} = S(x_n)$ $n \in \mathbb{N}$.
Show that there exists some $N \in \mathbb{N}$ such that $\forall$ $m, n \geq N$, then $x_n = x_m$. [6]

(ii) Let $x_1 \in \mathbb{N}$; $x_{n+1} = S\left(S(x_n) + x_n\right)$; $n \in \mathbb{N}$
Calculate x_{1984} if $x_1 = 1983$. [5]

(b) Prove that $\lim\limits_{n\to\infty} x_n = a$ if and only if $\forall n \in \mathbb{N}$, $x_n = a + \alpha_n$ where $\{\alpha_n\}$ is a sequence which is infinitely small. [6]

(c) Prove that $\lim\limits_{n\to\infty} x_n = 1$, where $x_n = \dfrac{n}{n+1}$. [3]

(d) Suppose that $\lim\limits_{n\to\infty} x_n = 0$ and $x_n > -1$ $\forall n \in \mathbb{N}$.
If $p \in \mathbb{N}$, prove that $\lim\limits_{n\to\infty} \sqrt[p]{1 + x_n} = 1$. [5]

B6. (a) Define the upper limit and lower limit of a sequence $\{X_n\}$. [4]

(b) State the Bolzano-Weierstrass theorem. [2]

(c) Use the Weierstrass theorem to prove that:

(i) the sequence $\{x_n\}$, where $x_1 = 0$; $x_{n+1} = \sqrt{6 + x_n}$; $(n \in \mathbb{N})$; has a finite limit. [5]
Calculate this limit. [3]

(ii) the sequence $\{x_n\}$ where $x_n = \left(1 + \dfrac{k}{n}\right)^n$ has a finite limit. [4]
Calculate this limit. [2]
Hint: Use the fact that $\lim\limits_{p\to\infty} \left(1 + \dfrac{1}{p}\right)^p = e$.

(d) Calculate the upper and lower limits of $x_n = \dfrac{n^2 \sin\left(\frac{n\pi}{2}\right) + 1}{n+1}$. [5]

B7. (a) State the Cauchy condition for convergence of a sequence. [2]

(b) Prove that the sequence $x_n = \sum_{k=1}^{n} \frac{1}{k}$ $(n \in \mathbb{N})$ diverges. [5]

(c) Calculate the following limits:

(i) $\lim_{n\to\infty}\left(\frac{2^n+1}{2^n}\right)^{2^n}$. [3]

(ii) $\lim_{n\to\infty}\left(\frac{2002}{n}\right)^n$ [3]

(iii) $\lim_{n\to\infty}\sqrt{n^2+n}-\sqrt{n^2-n}$. [3]

(iv) $\lim_{n\to\infty}\frac{10^n+n!}{2^n+(n+1)!}$. [3]

(d) Use the comparison test to determine whether $\sum_{n=1}^{\infty}\frac{1}{2n-1}$ converges or diverges. [6]

B8. (a) State the following convergence tests for series:

(i) D'Alembert's criterion [2]

(ii) Cauchy's integral test [2]

(iii) Leibniz's convergence test. [2]

(b) Investigate the convergence of:

(i) $\sum_{n=1}^{\infty}\frac{1}{n!}$ [3]

(ii) $\sum_{n=1}^{\infty}\left(\frac{3n}{3n+1}\right)^n$ [3]

(iii) $\sum_{n=1}^{\infty}\frac{3^n n!}{n^n}$ [3]

(iv) $\sum_{n=1}^{\infty}\frac{2n+1}{(n+1)^2(n+2)^2}$ [3]

(c) Prove that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n}$ converges uniformly in all its convergence interval $]-1,1[$ [4]

(d) Expand $f(x) = \cosh x$ in power series of x. Find the interval of convergence of the series and use the Lagrange theorem to find its remainder. [3]

END OF QUESTION PAPER