NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 1202

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA1202: REAL ANALYSIS

MAY 2005

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Sections B.

SECTION A: Answer ALL questions in this section [40].

A1. (a) State the following convergence tests for series:

(i) D'Alembert's Ratio criterion. [2]

(ii) Cauchy's Integral test. [2]

(iii) Leibniz's Convergence test. [2]

A2. Find the values of p for which the integral

$\int_1^\infty \frac{dx}{x(\ln x)^p}$ converges. [5]

A3. Use the Squeeze Theorem to establish that

$\lim_{x\to 0} x^2 \sin^2\left(\frac{1}{x}\right) = 0.$ [4]

A4. (a) Show that $\lim_{x\to 5} \frac{4x+7}{x^2-25}$ does not exist. [2]

(b) Find the interval of convergence for the power series

$\sum_{k=1}^{\infty} \frac{(-3)^k}{(k+1)(k+2)}(x-1)^k$. State the radius of convergence. [6]

A5. For approximately what values of x can one replace $\sin 2x$ by $2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!}$ with an error of magnitude no more than 5×10^{-6}? [7]

A6. Let $\{x_k\}$ be a Cauchy sequence. Prove that if a sequence is Cauchy, then it is bounded. [5]

A7. Let f be defined on $[0, 1]$ by

$$f(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Show that f is Riemann integrable on $[0, 1]$ and evaluate $\int_0^1 f(x)dx$. [5]

SECTION B: Answer THREE questions in this section [60].

B8. (a) Define a Cauchy sequence. [2]

(b) For $k \in N$, let $x_k = \sum_{j=0}^{k} \frac{(-1)^j}{2j!}$. We claim that x_k is Cauchy. Prove the claim. Compute an approximation, accurate to five decimal places, of the value of its limit. [10]

(c) State and prove the Uniqueness Theorem for convergence of a sequence $\{a_n\}$. [8]

B9. (a) Show that the Maclaurin's series for $\cos 2x$ converges to $\cos 2x$ for every value of x. [**Hint:** use Taylor's Remainder with $M = 1, r = 2$ and $a = 0$]. [4]

(b) State the Bolzano-Weierstrass Theorem. [2]

(c) Show by induction that, in the proof of the Bolzano-Weierstrass Theorem, the length of the interval I_k is $\frac{(b_0 - a_0)}{2^k}$. [4]

(d) In the proof of the Bolzano-Weierstrass Theorem, for every k and m with $m \geq k$, we have

$$a_0 \leq a_k \leq a_m \leq b_m \leq b_k \leq b_0.$$

Let $A = \{a_k : k \in N\}$ be the set of all left endpoints of the intervals I_k and let $B = \{b_k : k \in N\}$ be the set of all right endpoints. A is bounded above by b_0 and B is bounded below by a_0. By the Completeness Axiom it follows that $\alpha = \sup A$ and $\beta = \inf B$ exist in $\Re$ .We claim that $\alpha = \beta$.

(i) Show that $\alpha \leq \beta$, by using a "slick canonical proof". [3]

(ii) Show that α and β are actually equal. [7]

B10. (a) Define supremum and infimum for, S, a nonempty subset of $\Re$. [4]

(b) Consider the open interval
$S = (-4, 0) = \{x \in \Re : -4 < x < 0\}$
We claim that $\inf S = -4$ and $\sup S = 0$, prove the claim. [12]

(c) Define a monotonic sequence, hence identify each of the following sequences as monotonically increasing, decreasing, both increasing and decreasing, or not monotonic:

(i) $\{1\}$,

(ii) $\left\{\frac{1}{n}\right\}$,

(iii) $\{(-1)^n\}$. [4]

B11. (a) Define the Riemann integral for, f, a bounded, real-valued function on the interval $[a, b]$. [5]

(b) Suppose that f is a bounded function on the interval $[a, b]$ and that c is any point in $[a, b]$.
Show that if both the integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ exist, then $\int_a^b f(x)dx$ also exists and $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$. [15]

B12. (a) Explain what the statement "$\sum_{k=1}^{\infty} a_k$ converges" means. [2]

(b) Test the following series for convergence:

(i) $\sum_{k=0}^{\infty} \frac{2^k + 500}{3^k}$. [2]

(ii) $\sum_{k=1}^{\infty} \frac{1}{k^p}$, (p any number whose range has to be determined for convergence and divergence of the series). [5]

(c) Show that $\sum_{k=1}^{\infty} \frac{\sin kx}{k^2}$ converges for all real values of x. [3]

(d) Test for absolute or conditional convergence:

(i) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+3}$. [3]

(ii) $\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{n^2+1}$. [3]

(iii) $\sum_{n=1}^{\infty} n^2 \left(\frac{2}{3}\right)^n$. [2]

END OF QUESTION PAPER