NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 1203

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

ORDINARY DIFFERENTIAL EQUATIONS

DECEMBER 2002

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A: Answer ALL questions in this section [40].

A1. It is assumed that the earth cannot support a population greater than 20 billion persons, and that the rate of population growth is proportional to the difference between how close the world population is to this limiting value. What is the mathematical expression describing the world population as a function of time? [4]

A2. Find the general solution of

$$3y + e^t + (3t + \cos y)\frac{dy}{dt} = 0.$$

[6]

A3. By setting $v(x) = x + 2y$, find the general solution of the equation

$$\frac{dy}{dx} = \frac{x + 2y + 1}{x + 2y}.$$

[5]

A4. Prove that if $a \in \mathbb{R}$, then $e^{at}\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\{f(s-a)\}$. [6]

A5. Solve the following ordinary differential equation:

$$y' = \frac{2y}{x} + x^2 e^x.$$

[5]

A6. Find the general solution to $y'' - 4y' + 3y = 20e^{2t}\cos 3t$. [6]

A7. Show that t, t^{-1}, and t^2 are linearly independent solutions of the equation

$$t^3y''' + t^2y'' - 2ty' + 2y = 0.$$

Find the solution that satisfies the initial conditions

$$y(1) = 2, \quad y'(1) = 5, \quad y''(1) = 0.$$

[8]

SECTION B: Answer ANY THREE questions in this section [60].

B8. (a) If the population of the earth was found to be 3.5 billion in 1970 and is increasing at a rate of 2% per year, when will a population of 50 billion be reached? [5]

(b) Solve the initial value problem

$$y' = 6\cos^2 x - y\cot x, \quad y(\pi/4) = 3.$$

[6]

(c) Find the general solution of

$$y'' + \lambda^2 y = \kappa\cos\omega t, \quad \kappa \neq 0$$

when

(i) $\omega \neq \lambda$

(ii) $\omega = \lambda$

and discuss the boundedness of each of the solutions. [9]

B9. (a) Prove that if $y(x)$ is a continuous function, then it is a solution to the initial value problem $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$ if and only if

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt.$$

[5]

(b) Solve the initial value problem

$$y^2 - 2x + (2xy^2 + 2xy - 2x^2)\frac{dy}{dx} = 0, \quad y(1) = 1.$$

[6]

(c) Find the general solution of

$$x^2y'' - 6xy' + 10y = 0$$

by setting $y = x^k$. Hence use the method of variation of parameters to find the general solution of

$$x^2y'' - 6xy' + 10y = 3t^4 + 6t^3.$$

[9]

B10. (a) Prove that if **z** is a complex solution of $\frac{d\mathbf{z}}{dt} = A\mathbf{z}$, then the real and imaginary parts of **z** are real solutions. [8]

(b) Find the general solution of the system

$$\frac{dx}{dt} = -4x + 2y$$
$$\frac{dy}{dt} = -x - y$$

and sketch the resulting phase portrait. Find the solution satisfying

$$x(0) = 4, \quad y(0) = 3.$$

[12]

B11. (a) Find the inverse transform of

$$\frac{s}{(s^2 + 6s + 13)^2}.$$

[6]

(b) State and prove the Convolution Theorem. [8]

(c) Use the Convolution Theorem to evaluate

$$\mathcal{L}^{-1}\{\frac{1}{(s^2 + 1)(s^2 + 4)}\}.$$

[6]