NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 1204

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

ORDINARY DIFFERENTIAL EQUATIONS I

MAY/JUNE 2001

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A: Answer ALL questions in this section [40].

A1. A certain radioactive isotope is present in a sample of carbon. Let N be the number of atoms of the isotope and C denote the number of carbon atoms present. Express the following statement in terms of a differential equation (do not solve it):
"The rate of decay of the isotope is proportional to the number of atoms of it present and inversely proportional to the square of the number of carbon atoms present." [2]

A2. Solve the ordinary differential equation

$$(x^2 - 3y^2)dx + 2xydy = 0$$

for $y \geq x \geq 0$. [6]

A3. Solve the following initial value problem by applying the principle of undetermined coefficients:

$$y'' + 6y' + 9y = 27x^2, \quad y(0) = 2, \quad y'(0) = 1.$$

[6]

A4. Find the first four terms for the initial value problem

$$\frac{dy}{dx} = x + y^2, \quad y(0) = 0$$

by using the Picard iterative scheme. [6]

A5. Solve the initial value problem

$$y''' - 2y'' + y = 0, \quad y(0) = y'(0) = 0, \ y''(0) = 1.$$

[6]

A6. Use step function notation to re-write the following functions, and hence find the Laplace transform of each function:

(a) $f(t) = |2t - 1|$

(b) $g(t) = \begin{cases} 2 & \text{if} \quad t < 1 \\ 3t & \text{if} \quad 1 \leq t < 2 \\ 5 & \text{if} \quad 2 \geq 2 \end{cases}$

[4,5]

SECTION B: Answer THREE questions in this section [60].

B7. (a) State the convolution theorem, and hence find

(i) $\mathcal{L}^{-1}\left[\frac{1}{(s^2-5s+6)^2}\right]$

(ii) $\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right]$

[2,3,4]

(b) Solve the initial value problem by means of the Laplace transform:

$$(D^4 - 1)x = \begin{cases} 2 \ , & 0 \leq t < 3 \\ 0 \ , & t \geq 3 \end{cases} \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = 2.$$

[11]

SECTION B: Answer THREE questions in this section [60].

B8. (a) Show that the general solution of the corresponding homogeneous equation of the first order inhomogeneous linear differential equation

$$y' + p(x)y = q(x)$$

is $y = Ce^{-\int p(x)dx}$.
Considering C as a function of x show, by substituting back into the inhomogeneous equation, that $C(x)$ is the solution of

$$\frac{dC}{dx} = q(x)e^{\int p(x)dx},$$

in which the variables are separable.
Integrate this result and find the general solution of the inhomogeneous differential equation.

(b) Use the above analysis to solve the differential equation

$$y' = y\tan x + \cos x.$$

B9. (a) Show that the equation of the type

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

can be solved by the variables separable technique if we make the substitution $(y/x) = v$ i.e. $y = vx$.
Use the method to solve the initial value problem

$$\frac{dy}{dx} = \frac{y}{x} + \frac{2x^3\cos x^2}{y}, \quad y\sqrt{\pi} = 0.$$

[8]

(b) Solve the following initial value problem:

$$\frac{dy}{dx} = -y\tan x + \sec x, \quad y(0) = 0$$

[6]

(c) Radium decays exponentially and has a half-life of approximately 1600 years. Find a formula for the amount $N(t)$ remaining from $50mg$ of pure radium after t years. When will there be $20mg$ left? [6]

B10. (a) Solve the following initial value problems by means of the Laplace transform:

(i) $y'' + 2y' + y = f(t)$, $y(0) = 1$, $y'(0) = 0$, where

$$f(t) = \begin{cases} 1 \ , & 0 \leq t < 1 \\ 0 \ , & t \geq 1 \end{cases}$$

(ii) $y'' + 2y' + y = \delta(t) + u(t)$; $y(0) = 0$, $y'(0) = 1$.

(b) Express the initial value problem

$$y'' + 2y' + 2y = \sin \alpha t; \quad y(0) = y'(0) = 0$$

in terms of a convolution integral. [20]

B11. (a) By the method of variation of parameters, find the general solution of the equation:

$$y'' + 4y' + 4y = \frac{e^{-2x}}{x^2}.$$

[8]

(b) Find the general solution of the system

$$\frac{dx}{dt} = -4x + 2y$$
$$\frac{dy}{dt} = -x - y$$

and sketch the resulting phase portrait. Find the solution satisfying

$$x(0) = 4, \quad y(0) = 3.$$

[12]

END OF QUESTION PAPER