NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

May 2002 EXAMINATION

SMA1204 ORDINARY DIFFERENTIAL EQUATIONS

3 Hours

Answer ALL questions in Section A and any FOUR questions in Section B.

SECTION A : Answer ALL questions from this section.

[32 Marks]

1. Solve the following initial value problems:

   (a) $x\frac{dy}{dx} - y = x^2 \sin x, \quad y(\pi/2) = 1$

   (b) $(1+x^2)\frac{dy}{dx} = (1+y^2), \quad y(0) = -1$

[3+3 Marks]

2. Carry out three steps of the Picard Iteration scheme for the initial value problem

$$\frac{dy}{dx} = -y + x, \quad y(0) = 1.$$

[4 Marks]

3. Find the general solutions of

   (a) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

   (b) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 3\sin(2x)$

   (c) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 3e^{-2x}.$

[2+4+4 Marks]

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4. Find the general solution of the following system and sketch its phase portrait.

$$\frac{dx}{dt} = -4x + 2y$$
$$\frac{dy}{dt} = -x - y$$

Find also the solution satisfying $x(0) = 2, y(0) = 1$.

[6 Marks]

5. Use a power series method to find the first 5 non-zero terms in the series solution in the neighbourhood of $x = 0$ to the initial value problem

$$(1 - x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$$

[6 Marks]

SECTION B - Answer any FOUR questions from this section.

[68 Marks]

6. (a) Show that the following differential equation is exact and find its general solution.

$$(y\cos x + 4x) + (\sin x + 2y)\frac{dy}{dx} = 0$$

(b) By finding an integrating factor, solve the initial value problem

$$(3x^2 + 8y)\frac{dy}{dx} + 2xy = 0, \quad y(0) = 1.$$

(c) By making the substitution $z = y^{-2}$, solve the equation

$$x^2\frac{dy}{dx} + xy = y^3.$$

(d) By making the substitution $z = e^y$ solve the equation

$$(x^2 + xe^y)\frac{dy}{dx} = e^y + x^2e^{-y}.$$

[4+5+4+4 Marks]

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7. (a) Use the method of reduction of order to solve the following differential equations.

i. $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

ii. $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = -x^3 \sin x, \quad y(0) = y'(0) = 0.$

(b) By looking for solutions of the form $y = x^r$, find the general solution of the differential equation

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 0.$$

Hence use the method of variation of parameters to solve the differential equation

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = x^2e^x.$$

[4+5+8 Marks]

8. (a) Show that $\mathbf{x} = \mathbf{e}^{\lambda t}\mathbf{v}$ is a solution of the system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$, where λ and $\mathbf{v}$ are an eigenvalue and a corresponding eigenvector of $\mathbf{v}$.

(b) Find and classify the equilibrium point of the system

$$\begin{aligned}\frac{dx}{dt} &= 3x - 4y + 7\\ \frac{dy}{dt} &= x + y - 7\end{aligned}$$

(c) Find the general solution and sketch the phase portrait of the system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -1 & -2 \\ -4 & 1 \end{pmatrix}\mathbf{x}.$$

(d) Find the general solution of the system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -4 & 6 \\ -3 & 2 \end{pmatrix}\mathbf{x}.$$

[3+5+5+4 Marks]

9. (a) Verify that $y = \sin x^2$ and $y = \cos x^2$ are linearly independent solutions of the differential equation

$$2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 8x^2y = 0.$$

Hence find the particular solution of the differential equation, given that $y(0) = 1$ and that $y(\sqrt{\pi/2}) = 2$.

(b) Solve the following differential equations.

i. $\frac{d^2y}{dx^2} + 4y = 2\sin 2x$

ii. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 2x^2$

iii. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = e^{-x}\sin x$

[4+4+4+5 Marks]

10. (a) Use the method of Frobenius in the neighbourhood of $x = 0$ to find one solution of the differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - xy = 0.$$

(b) Write your solution above in terms of a Bessel function of integer order,

$$J_n(x) = \Sigma_{j=0}^{\infty}\frac{(-x)^{n+2j}}{j!(n+j)!2^{n+2j}}.$$

(c) Find and classify the singular points of the differential equation

$$(1-x^2)x^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} - (1-x^2)y = 0.$$

[10+4+3 Marks]

END OF EXAMINATION PAPER
