NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 1204

DEPARTMENT OF APPLIED MATHEMATICS

SMA1204 ORDINARY DIFFERENTIAL EQUATIONS

MAY 2005

Time : 3 hours

This paper contains TWO sections. Answer ALL questions in section A and THREE questions from section B.

SECTION A: Answer ALL questions in this section [40].

A1. Solve the homogeneous equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}.$$

[4]

A2. Find the integrating factor, $I(x)$, of the first order linear differential equation

$$y' + P(x)y = Q(x).$$

[4]

Hence solve the differential equation

$$y' + \frac{2}{10 + 2x}y = 4$$

given that $y(2) = 100$. [5]

A3. (a) State the condition for a set of functions $y_1(x), y_2(x), ..., y_n(x)$ to be a solution of the homogeneous differential

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + ... + a_0(x)y = 0.$$

Hence write down the general solution of the differential equation. [2]

(b) Given that e^x, e^{2x} and e^{3x} are solutions of

$$y''' - 6y'' + 11y' - 6y = 0,$$

show that the general solution of the differential equation is

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x}.$$

[4]

A4. At a school with 500 pupils it was found out that 5 pupils were infected with smallpox. It was assumed that the rate of change in the infected pupils is proportional to the product of the number of pupils who have the disease with the number that are disease free.

(a) By letting $N(t)$ denote the number of pupils with smallpox at time, t weeks show that

$$\frac{dN}{dt} = kN(500 - N)$$

where k is a positive constant. [2]

(b) Find the general solution of the equation in (a). [4]

(c) Given that after 2 weeks 20 pupils had contracted the disease how long will it take for half of the pupils to contract the disease? [5]

A5. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 9 & 1 \end{pmatrix}.$$

[5]

Hence or otherwise solve the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= 9x + y. \end{aligned}$$

[2]

Draw the phase portraits for the system. [3]

SECTION B: Answer any THREE questions in this section [60].

B6. (a) Find the general solution of

$$y''' - 2y'' - 5y' + 6y = (e^{2x} + 3)^2.$$

[10]

(b) Solve the system

$$\begin{aligned} \frac{dx}{dt} + 2x + 3y &= 0 \\ \frac{dy}{dt} + 3x + 2y &= 7e^{2t}. \end{aligned}$$

[10]

B7. (a) Solve the differential equation

$$(2x^2t - 2x^3)dt + (4x^3 - 6x^2t + 2xt^2)dx = 0.$$

[5]

(b) For the nonhomogeneous differential equation

$$(x^2 + 4)y'' + xy = x + 2,$$

(i) find a recurrence formula for the power series solution around $x = 0$ for the differential equations, [9]

(ii) hence find the general solution near $x = 0$. [6]

B8. Consider the differential equation $x^2y'' + (x^2 - 2x)y' + 2y = 0$.

(a) Determine whether $x = 0$ is a regular singular point of the differential equation. [2]

(b) Use the method of Frobenius to find one solution near $x = 0$. [10]

(c) Find the general solution near $x = 0$ of the differential equation. [8]

B9. (a) Use the method of variation of parameters to solve the differential equation

$$y'' - 2y' + y = \frac{e^x}{x}.$$

[8]

(b) Consider the Legendre equation

$$(1-x^2)y'' - 2xy + k(k+1)y = 0$$

for any positive integer k.

(i) Determine whether $x = 0$ or $x = 1$ is an ordinary point of the Legendre equation

$$(1-x^2)y'' - 2xy + k(k+1)y = 0$$

for any positive integer k. [2]

(ii) Find a recurrence formula for the power series solution around $x = 0$ for the Legendre equation. [6]

(iii) Show that whenever k is a positive integer, one solution near $x = 0$ of Legendre's equation is a polynomial of degree k. [4]

END OF QUESTION PAPER