# NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

## DEPARTMENT OF APPLIED MATHEMATICS

## JULY 2001 SUPPLEMENTARY EXAMINATION

## SMA 2101 VECTOR CALCULUS

3 hours
July 2001

2 pages

Answer ALL questions in section A and any THREE in Section B

Section A. Answer ALL questions [ 40 marks]

1. Obtain the solution of the equation

$$\mathbf{x} \times (1,1,1) = (2, -4, 2)$$

**[ 5 marks]**

2. Calculate the unit tangent and the unit normal to the curve defined by

$$\mathbf{x} = ( e^{t}, e^{-t}, t\sqrt{2} )$$

**[ 6 marks]**

3. If $\mathbf{f} = (x_1, x_2, x_3)/(x_1^2 + x_2^2 + x_3^2)$, find div **f** and curl **f**.

**[6 marks ]**

4. Parabolic coordinates (u, v, w ) are defined in terms of rectangular Cartesian coordinates $(x_1, x_2, x_3)$ by $x_1 = (u^2 - v^2)/2, x_2 = uv, x_3 = w.$ Calculate the base vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and verify that they are mutually orthogonal.

**[ 8 marks]**

5. Show that the function $\phi = x_1 x_2$ has a saddle point at the origin.

**[ 5 marks]**

6. The transformation from rectangular Cartesian coordinates to polar coordinates is given by

$$x_1 = r\cos\theta, x_2 = r\sin\theta$$

show that the Jacobian is r.
Evaluate

$$\iint_R \exp[-(x_1^2 + x_2^2)]dx_1 dx_2$$

where R is the positive quadrant $(x_1 \geq 0, x_2 \geq 0)$.

Hence deduce that $\int_0^\infty \exp(-x^2)dx = \frac{\sqrt{\pi}}{2}$

**[10 marks]**

**SECTION B.** Answer any THREE questions [60 marks]

7. The three unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ form a basis, and $\alpha_1$, $\alpha_2$, $\alpha_3$ are the angles between $\mathbf{e}_2$ and $\mathbf{e}_3$, $\mathbf{e}_3$ and $\mathbf{e}_1$, $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively.

   Prove that $\mathbf{u} = \mathbf{e}_2 - (\mathbf{e}_1 \cdot \mathbf{e}_2)\,\mathbf{e}_1$ is perpendicular to $\mathbf{e}_1$ and find its magnitude.

   If $\mathbf{v} = \mathbf{e}_3 - (\mathbf{e}_1 \cdot \mathbf{e}_3)\,\mathbf{e}_1$, prove that $\cos\alpha_1 = \cos\alpha_2 \cos\alpha_3 + \sin\alpha_2 \sin\alpha_3 \cos\beta$ where $\beta$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

   Let $\mathbf{q}_1 = \mathbf{e}_1$, $\mathbf{q}_2$ be a unit vector parallel to $\mathbf{u}$ and $\mathbf{w} = \mathbf{e}_3 + a_1\mathbf{q}_1 + a_2\mathbf{q}_2$, determine $a_1$ and $a_2$ such that $\mathbf{w}$ is perpendicular to both $\mathbf{q}_1$ and $\mathbf{q}_2$. Hence, define an orthonormal basis $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$.

   **[ 20 marks]**

8. Find the mass of a tetrahedron with vertices (0, 0, 0), (1, 0, 0), ( 0,1,0) and ( 0,0,1) if the density of the material $\rho = x_1 x_2$.

   **[ 20 marks]**

9. State and prove the divergence ( Gauss) theorem.

   Evaluate the surface integral $\iint_S \mathbf{x} \cdot \mathbf{n}\, dS$, where $\mathbf{x}$ and $\mathbf{n}$ are the position vector and unit outward normal to the closed surface S respectively. Give your answer in term of the volume enclosed by S.

   **[ 20 marks ]**

10. If $\mathbf{f} = \left(x_1^2 x_2, x_3, 0\right)$ and S is the hemisphere $x_1^2 + x_2^2 + x_3^2 = a^2$, $x_3 \geq 0$, verify by carrying out the integrals on both sides that

$$\iint_S \text{curl}\,\mathbf{f} \cdot \mathbf{n}\, dS = \oint_C \mathbf{f} \cdot d\mathbf{x}$$

   where C is the curve that encloses S and $\mathbf{n}$ is the unit outward normal to S.

   **[ 20 marks ]**

**END**