NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA2102 ADVANCED LINEAR ALGEBRA

December 2004
3 Hours

Answer ALL questions from this Section A and any FOUR questions in Section B

SECTION A : Answer ALL questions from this section

1. Explain carefully what is meant by saying that the set $\{v_1, \ldots, v_n\}$

   (i) is linearly independent.
   (ii) forms a basis for a vector space V.

[4 marks]

2. (a) State a theorem concerning the dimensions of the kernel and image of a linear transformation.

   (b) Suppose that T: $\mathbf{R}^3 \to \mathbf{R}^7$ is linear. What are the possible dimensions of the kernel of T? What would be the possible dimensions of kernel T if T: $\mathbf{R}^5 \to \mathbf{R}^3$?

   (c) Let T be a linear map. Prove that ker T = {0} if and only if T is one-to-one.

[2 + 2 + 4 = 8 marks]

3. (a) A square matrix A is symmetric if $A = A^T$. Let A be any square matrix. Show that AA^T is symmetric.

   (b) Let $A = (a_{ij})$, $B = (b_{jk})$ and $C = (c_{kl})$. Prove that (AB)C = A(BC).

[2 + 3 = 5 marks]

4. Show that the set of points $\{(x,y,z) \in \mathbf{R}^3 : x + 2y - z = 0\}$ is a subspace of R^3.

[4 marks]

5. Let T: $\mathbf{R}^3 \to \mathbf{R}^2$ be defined by $T(x,y,z) = (x-2y+z, 2x+y-z)$.

(a) Find the matrix of T relative to the standard bases.

(b) Show that B={(1,1,0) , (1,0,-1) , (0,2,1)} is a basis for $\mathbf{R}^3$ and find the co-ordinates of (1,1,1) relative to this basis.

(c) Find the matrix of T relative to the bases B for $\mathbf{R}^3$ and C = {(1,2) , (-1,1)} for $\mathbf{R}^2$.

[2 + 4 + 4 = 10 marks]

6. Let U and W be distinct 4 – dimensional subspaces of a vector space V of dimension 6. Find the possible dimensions of $U \cap W$.

[5 marks]

Section B: Answer any FOUR questions from this section

7. (a) Let V be the vector space of n-square matrices of K. Let M be an arbitrary matrix in V. $T: V \to V$ is defined by

$$T(A) = AM + MA, \quad \text{where } A \in V.$$

Show that T is linear.

[3 marks]

(b) Find whether the following systems have non-trivial solutions. If so, find them.

(i)
$$\begin{aligned} x_1 + 2x_2 + x_3 - x_4 &= 0 \\ 2x_1 - x_2 - 4x_3 - 6x_4 &= 0 \\ x_1 - x_2 - 2x_3 - 4x_4 &= 0 \end{aligned}$$
,

(ii)
$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 0 \\ 3x_1 + x_2 + x_3 &= 0. \end{aligned}$$

[6 marks]

(c) Solve, if possible, the following systems, or else show that no solution exists:

(i)
$$\begin{aligned} x_1 + x_2 - 2x_3 + x_4 + 3x_5 &= 1 \\ 2x_1 - x_2 + 2x_3 + 2x_4 + 6x_5 &= 2 \\ 3x_1 + 2x_2 - 4x_3 - 3x_4 - 9x_5 &= 3, \end{aligned}$$

(ii)
$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 5 \\ 2x_1 + 3x_2 - x_3 - 2x_4 &= 2 \\ 4x_1 + 5x_2 + 3x_3 &= 7. \end{aligned}$$

[7 marks]

8. (a) Let V and W be vector spaces and $T: V \to W$ be linear and $w \in W$. Suppose $v_0 \in V$ such that $Tv_0 = w$. Show that any solution x of the equation $Tx = w$ is of the form $x = v_0 + u$ where $u \in \ker T$.

[5 marks]

(b) Let V and W be vector spaces and $T: V \to W$ be a linear transformation.

Prove that (i) ker T is a subspace of V.
(ii) im T is a subspace of W.

[6 marks]

(c) Let $T: \mathbf{R}^3 \to \mathbf{R}^3$ by $T(x,y,z) = (3x - y, x - y + z, -x + 2y - z)$.

Show that T^{-1} exists and find it.

[5 marks]

9. (a) Find the characteristic polynomial, eigenvalues and eigenspaces of

$$\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

(b) Diagonalize the following matrices if possible, or else explain why they are not diagonable.

(i) $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$, (ii) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{pmatrix}$.

[8 + 8 = 16 marks]

9. (a) Give a direct proof of the Cayley- Hamilton theorem for 2 x 2 matrices.

(b) Find the minimal polynomial of

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ a_0 & a_1 & a_2 \end{pmatrix}.$$

(c) Find an orthogonal matrix U such that U^TAU is diagonable where

$$A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{pmatrix}$$

and hence diagonalize A.

[5 + 3 + 8 = 16 marks]

11. (a) Find an orthonormal basis for the subspace of $\mathbf{R}^4$ generated by (1,1,0,0) (1,-1,1,1) and (-1, 0,2,1).

(b) Find invertible matrices P and Q such that PAQ is the normal form of A where

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 0 & -1 & 2 \\ 0 & 2 & -4 \end{pmatrix}.$$

[6 + 10 = 16 marks]

END OF QUESTION PAPER