NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 2102

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

ADVANCED LINEAR ALGEBRA

DECEMBER 2005

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A: Answer ALL questions in this section [40].

A1. Let F be a field. Recall that if $x \in F$ then the negative (additive inverse) of x is unique and that $x0 = 0$. Now prove that

$$(\forall a, b \in F) \quad (-a)b = a(-b) = -(ab)$$

and deduce that

$$(\forall a, b \in F) \quad (-a)(-b) = ab.$$

[3+2]

A2. Suppose A is invertible and, say, it is row reducible to the identity matrix I by the sequence of elementary operations $e_1, ..., e_n$.

(a) Show that this sequence of elementary row operations applied to I yields A^{-1}.

(b) Use this result to obtain the inverse of

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$$

[2+4]

A3. (a) A square matrix A is called skew-symmetric if $A = -A^T$. Let A be any square matrix. Show that $A - A^T$ is skew symmetric.

(b) Let $A = (a_{ij}), B = (b_{jk})$ and $C = (c_{kl})$. Prove that $(AB)C = A(BC)$.

[2+3]

A4. (a) Explain what is meant by the terms **kernel** and **image** of a linear map.

(b) Let T be a linear map. Prove that $\ker T = \{0\}$ if and only if T is one-to-one.

(c) State a theorem relating the dimensions of the image and kernel of a linear map.

(d) Let $T : V \to V$ be linear, where V is a finite dimensional vector space. Show that T is one-to-one if and only if it is onto.

[2+4+1+3]

A5.

Let

$$\begin{aligned} V &= \left\{(x_1, x_2, x_3, x_4) \in R^4 | x_1 + x_2 + x_3 = 0\right\} \\ W &= \left\{x_1, x_2, x_3, x_4) \in R^4 | x_3 = -x_4\right\} \end{aligned}$$

Find bases for $V \cap W$ and $V + W$.

[5]

A6. Let $T : R^3 \to R^2$ be defined by $T(x, y, z) = (x + 2y - z, 2x - y + z)$

(a) Find the matrix of T relative to the standard bases.

(b) Show that $\{(1, -1, 2), (1, 0, 1), (2, 0, 0)\}$ and $\{(1, 2), (1, 0)\}$ are the bases for R^3 and R^2 respectively and find the matrix of T relative to these bases.

[9]

SECTION B: Answer THREE questions in this section [60].

B7. (a) Let F be the vector space of all functions $f : R \to R$ and let V be the subspace of all differentiable functions. Show that differentiation is a linear transformation of V into F.

(b) Find all the solutions of the system

$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 + x_5 &= 0 \\ 2x_1 + x_2 + 4x_3 - x_5 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 - 4x_2 + 11x_3 - 9x_4 - 5x_5 &= 0 \end{aligned}$$

(c) Find whether the following systems have non-trivial solutions. If so, find them.

$$(i)\quad \begin{aligned} x_1 - x_2 - x_3 + 2x_4 &= 0 \\ 2x_1 - x_2 + x_3 + 3x_4 &= 0 \\ 3x_1 - 2x_2 + x_3 + 4x_4 &= 0 \end{aligned} \qquad (ii)\quad \begin{aligned} x_1 - 2x_2 - x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 0 \\ 3x_1 + x_2 + x_3 &= 0 \end{aligned}$$

[5+7+8]

B8. (a) Let V and U be vector spaces and $T : V \to U$ be linear and $u \in U$. Suppose $v_0 \in V$ such that $Tv_0 = u$. Show that any solution of x of the equation $Tx = u$ is of the form $x = v_0 + w$ where $w \in kerT$. [5]

(b) Let V and W be vector spaces and $T : V \to W$ be a linear transformation. Prove that

(i) ker$T$ is a subspace of V.

(ii) im$T$ is a subspace of W. [7]

(c) Let $T : R^3 \longrightarrow R^3$ be defined by $T(x, y, z) = (-x + 3y + 2z, 4z - y, 2z)$

(i) Show that T is invertible.

(ii) Find a formula for T^{-1}.

[8]

B9. (a) Find the eigenvalues and basis for the eigenspace of

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}.$$

(b) Diagonalize the following matrices if possible, or else explain why they are not diagonizable.

$$(i)\quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad (ii)\quad \begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}.$$

[9+11]

B10. (a) Give a direct proof of the Cayley-Hamilton theorem for 2×2 matrices.

(b) Show that a matrix A and its transpose have the same characteristic polynomial.

(c) Find an orthogonal matrix U such that $U^T AU$ is diagonable where

$$\begin{pmatrix} 7 & -1 & -2 \\ -1 & 7 & 2 \\ -2 & 2 & 10 \end{pmatrix}$$

and hence diagonalize A.

[6+4+10]

B11. (a) Let V be a real inner product space with $v \in V$ and let $v_1, ..., v_n$ be an orthonormal set. Let $c_i = (v, v_i)$. Show that $v - \sum c_i v_i$ is orthogonal to $\sum c_i v_i$.

(b) Find an orthonormal basis for the subspace of R^4 generated by $(1, 1, 0, 0)$ $(1, -1, 1, 1)$ and $(-1, 0, 2, 1)$.

(c) Find invertible matrices P and Q such that PAQ is the normal form of A where

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 1 & 2 & 1 & -1 \\ 2 & 9 & 4 & -5 \end{pmatrix}$$

[4+6+10]

END OF QUESTION PAPER