NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

AUGUST 2004 SUPPLEMENTARY EXAMINATIONS

SMA2103 THEORETICAL MECHANICS

Answer ALL Questions in Section A and THREE Questions in Section B

SECTION A: 28 Marks

1. Given that the position vector of a particle in terms of plane polar coordinates is given by $\vec{r} = re_r$. Show that the velocity and acceleration of the particle are given by:
   (a) $\vec{v} = \dot{r}e_r + r\dot{\theta}e_\theta$. **[2]**
   (b) $\vec{a} = (\ddot{r} - r\dot{\theta})e_r + \frac{e_\theta}{r}\frac{d}{dt}(r^2\dot{\theta})$. **[2]**

2. The position vector of a particle is given by
$$\vec{r} = a\cos(\omega t)\sin(\Omega t)\hat{\imath} + a\sin(\omega t)\sin(\Omega t)\hat{\jmath} + a\cos(\Omega t)\hat{k}$$
   (a) Show that the particle moves on a sphere of radius a.
   (b) Find the velocity of the particle and show that its magnitude is given by $a(\Omega^2 + \omega^2\sin^2(\Omega t))^{1/2}$. **[2; 2]**

3. A particle is projected with speed U_0 from a point on a plane inclined at an angle α to the horizontal. If the angle of projection with respect to the plane is θ
   (a) Show that the range up the plane is
$$R = \frac{2U_0^2\sin\theta\cos(\theta+\alpha)}{g\cos\alpha}$$
   where g is the (constant) acceleration due to gravity.
   (b) What is the time of flight? **[2; 2]**

4. A particle of mass 2 moves in a force field depending on time given by
$$\vec{F} = 24t^2\hat{\imath} + (36t - 16)\hat{\jmath} - 12t\hat{k}$$
   Assuming that at $t = 0$ the particle is located at $\vec{r}_0 = 3\hat{\imath} - \hat{\jmath} + 4\hat{k}$ and has velocity $\vec{v}_0 = 6\hat{\imath} + 15\hat{\jmath} - 8\hat{k}$ find
   (a) the velocity, and
   (b) the position vector at any time t. **[1; 2]**

5. Consider a projectile problem in which the only force acting on the particle of mass m is the constant gravitational force. Supposing that the initial conditions are

$$r = 0, \quad \dot{\vec{r}} = u_0[\hat{\imath}\cos\theta + \hat{k}\sin\theta]$$

at time $t = 0$. Find the total work done by constant gravitational force during the flight of the projectile. **[4]**

6. A particle of mass m is suspended from a fixed support by a spring of natural length l and stiffness k. It moves along a vertical line through the point of suspension such that its speed is u_0 when it passes in a downward direction through the equilibrium position. Determine the tension in the spring. **[4]**

7. Consider N particles $P_1, P_2, P_3, \ldots, P_N$, where the particles P_i has mass m_i and position vector $\vec{r}_i$ relative to the origin O of an inertial frame of reference.

   (a) Show that the kinetic energy of the system is

$$E_k = 1/2m\dot{\vec{r}}_c \cdot \dot{\vec{r}}_c + 1/2\sum_{n=i}^{N}(\dot{\vec{r}}_i - \dot{\vec{r}}_c) \cdot (\dot{\vec{r}}_i - \dot{\vec{r}}_c)$$

   (b) Explain the significance of the two terms in the expression of the kinetic energy. **[4; 1]**

SECTION B: 72 Marks

8. (a) What do you understand by the term frame of reference?

   (b) For the position vector $\vec{r}$, the time derivative in an inertial frame (fixed frame, F) is obtained by adding the term $\omega \times \vec{r}$ to the time derivative in a rotating frame (moving frame, M), namely

$$(d\vec{r}/dt)_F = (d\vec{r}/dt)_M + \vec{\omega} \times \vec{r}$$

   where $\vec{r}$ is the angular velocity of the rotating frame. Find the relationship between the acceleration vectors in the two frames.

   (c) An xyz frame of reference is rotating with respect to an XYZ frame of reference, having the same origin and assumed to be fixed in space [i.e. it is an inertial frame]. The angular velocity of the xyz relative to the XYZ system is given by

$$\vec{\omega} = 2t\hat{\imath} - t^2\hat{\jmath} + (2t + 4)\hat{k}$$

   where t is the time. The position vector of a particle at time t as observed in the xyz system is given by

$$\vec{r} = (t^2 + 1)\hat{\imath} - 6t\hat{\jmath} + 4t^3\hat{k}$$

   (i) Find the apparent velocity and the true velocity at time $t = 1$ in the above.

(ii) Find the apparent acceleration and the true acceleration of the particle in the above problem.

(iii) Referring to the above information, find the coriolis acceleration, the centripetal acceleration and their magnitudes at time $t = 1$.**[24]**

9. (a) Write down Kepler's laws of planetary motion.

(b) A particle P_1 of mass M is fixed at the origin of an inertial frame of reference. A particle P_2 of mass m is free to move under the gravitational force of attraction. Let $\vec{r}$ denote the position vector of P_2 relative to P_1. The equation of motion for the particle P_2 is

$$\ddot{\vec{r}} = -\frac{GMme_r}{r^2} \qquad (1)$$

where $\vec{r} = re_r$.

(i) Using plane polar coordinates (r, θ) to describe the motion of the particle P_2 relative to P_1, show that

$$r^2\dot{\theta} = L_0$$

and

$$\ddot{r} - \frac{L_0^2}{r^3} = -\frac{GM}{r}$$

where L_0 is a constant and is the (initial) value of the component of the angular momentum vector per unit mass.

(ii) From the equation of motion above (1) show that the kinetic energy of the particle is given by

$$E_k = 1/2m\dot{\vec{r}} \cdot \dot{\vec{r}} = 1/2m(\dot{r}^2 + r^2\dot{\theta}^2) = 1/2m(\dot{r}^2 + L_0^2/r^2)$$

(c) From the equation of motion above

(i) Show that the path of the particle P_2 must satisfy the equation

$$\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - \frac{1}{r} = -\frac{GM}{L_0^2}$$

(ii) Choose a suitable transformation to transform this equation of motion into a second order differential equation

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{L_0^2}$$

which has constant coefficients.

(iii) If the initial conditions are $r = r_0$, $\dot{r} = 0$, $\theta = 0$, $\dot{\theta} = \frac{L_0}{r_0^2}$ at time $t = 0$, show that the solution of this equation is

$$\frac{l}{r} = e\cos(\theta - \theta_0) + 1.$$

Define the constants l and e. State what values must e assume in order for the path of particle P_2 to be an ellipse, a circle, a parabola or a hyperbola.

10. (a) State Chasle's Theorem about the general motion of a rigid body.

(b) Calculate the moment of inertia of a uniform rigid rod of mass m and length L about a perpendicular axes:

(i) Through the centre of mass

(ii) Through an end point.

(c) A given rigid body R, will in general have six degrees of freedom and its position at any time t, relative to a given inertial frame S, can be found using the two equations of motion:

$$\frac{d\vec{P}}{dt} = \vec{F} \text{ and } \frac{d\vec{L}}{dt} = \vec{M}$$

where $\vec{F}$ is the total external force acting on the body and $\vec{M}$ is the total moment of the external force.

(i) Write down in integral form the expressions for the total linear momentum $\vec{P}$ and the total angular momentum $\vec{M}$.

(ii) Show that the angular momentum $\vec{L}$ of the rigid body R relative to an origin fixed in the rigid body can be written in terms of the angular velocity $\vec{\omega}$ to become

$$\vec{L} = I\vec{\omega}$$

(iii) Show also that if the rigid body rotates with angular velocity $\vec{\omega}$, then the kinetic energy of rotation is given by

$$T = \frac{1}{2}I\vec{\omega}^2$$

where I is the moment of inertia about the origin. **[24]**

11. Given that in spherical coordinates the position vector of a particle is given by $\vec{r} = re_r$ where

$$e_r = \hat{\imath}\sin\theta\cos\phi + \hat{\jmath}\sin\theta\sin\phi + \hat{k}\cos\theta$$

$$e_\theta = \hat{\imath}\cos\theta\cos\phi + \hat{\jmath}\cos\theta\sin\phi - \hat{k}\sin\theta$$

$$e_\phi = -\hat{\imath}\sin\phi + \hat{\jmath}\cos\theta$$

(a) Show that

(i)

$$\frac{de_r}{dt} = \dot{\theta}e_\theta + \dot{\phi}e_\phi\sin\theta$$

(ii)

$$\frac{de_\theta}{dt} = -\dot{\theta}e_r + \dot{\phi}e_\phi \cos\theta$$

(iii)

$$\frac{de_\phi}{dt} = -\dot{\phi}e_r \sin\theta - \dot{\phi}e_\theta \cos\phi$$

(b) Show that

(i)

$$e_r \cdot e_\theta = 0 \quad \text{and that}$$

(ii)

$$e_r \cdot e_r = 1$$

(c) (i) Show that the path of a particle whose position vector at time t is given by $\vec{r} = a\cos\omega t\hat{i} + a\sin\omega t\hat{j}$, with a, ω constants is a circle.

(ii) Find the velocity and acceleration and show that $\vec{V} \cdot \vec{a} = 0$. **[24]**

12. (a) Use Lagrange's equations to set up the differential equation of the vibrating masses given that the kinetic energy of the system is

$$T = \frac{1}{2}m[\dot{x}_1^2 + \dot{x}_2^2]$$

and that the potential energy of the system is

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}(x_2 - x_1)^2 + \frac{1}{2}kx_2^2$$

(b) A particle of mass m moves in a conservative force field. Find

(i) The Lagrangian function,

(ii) The equations of motion in cylindrical coordinates

(c) Suppose that the forces acting on a system of particles are derived from a potential function V, that is, suppose that the system is conservative. Prove that if $L = T - V$ is the Lagrangian function, then

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{q}_\alpha}] - \frac{\partial L}{\partial q_\alpha} = 0$$

[24]