NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

DECEMBER 2004 EXAMINATIONS

SMA2103 THEORETICAL MECHANICS

TIME: 3 HOURS

Answer ALL **Questions in SECTION A and** THREE **Questions in SECTION B**

SECTION A: 28 MARKS

1. Determine the number of degrees of freedom in each of the following cases:

   (a) A rigid body moving parallel to a fixed plane, [2]

   (b) A system consisting of a thin rigid rod which can move freely in space and a particle which is constrained to move on the rod. [2]

2. Show that the velocity of a particle is independent of the origin to which its position vector is referred. [4]

3. (a) Show that the path of a particle whose position vector at time t is given by

   $\bar{r} = \hat{i}a\cos wt + \hat{j}a\sin wt$

   with a and w as constants, is a circle. [3]

   (b) Find the velocity and acceleration of the particle. [2]

   (c) Show that $\bar{V} \bullet \bar{a} = 0$ [2]

4. Given that the position vector of a particle in terms of plane polar coordinates is given by:

   $\bar{r} = re_r$ where $e_r = \cos\theta\hat{i} + \sin\theta\hat{j}$ and

   $e_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j}$

   Show that the velocity and acceleration of the particle are given by:

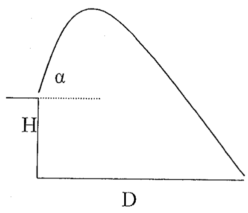
   (i) $\bar{V} = \dot{r}e_r + r\dot{\theta}e_\theta$ [2]

   (ii) $\bar{a} = \left(\ddot{r} - r\dot{\theta}^2\right)e_r + \frac{e_\theta}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$ [2]

5. A projectile is launched at an angle α from a cliff of height H above sea level. If it falls into the sea at a distance D from the base of the cliff, prove that its maximum height above sea level is

   $$H + \frac{D^2\tan^2\alpha}{4(H + D\tan\alpha)}$$ [5]

(see diagram below)



6. A train of mass m moves along a level track. The engine provides a tractive force $\bar{F}$ and works at a constant rate P_o. There is a constant resistive force of magnitude R_o opposing the motion.

   (i) Show that the equation governing the motion has the form:

   $m\dot{x}\ddot{x} = P_o - R_o\dot{x}$ (provided $\dot{x} \neq 0$) (*) [2]

   (ii) The maximum speed occurs when $\ddot{x} = 0$ and has a value $P_o/R_o = U_{max}$.

   By setting $\dot{x} = u$ and $\ddot{x} = \frac{du}{dt}$, rewrite equation (*) in the form:

$$\frac{m}{R_o}\left[\frac{P_o}{P_o - R_o u} - 1\right]\frac{du}{dt} = 1$$ [2]

SECTION B: 72 MARKS

7. Given that in spherical coordinates the position vector of a particle is given by $\vec{r} = re_r$,

   Where $e_r = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta$

   $e_\theta = \hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta$

   $e_\phi = -\hat{i}\sin\phi + \hat{j}\cos\phi$

   (a) Show that

   (i) $\frac{de_r}{dt} = \dot{\theta}e_\theta + \dot{\phi}e_\phi\sin\theta$ [4]

   (ii) $\frac{de_\theta}{dt} = -\dot{\theta}e_r + \dot{\phi}e_\phi\cos\theta$ [4]

(iii) $\frac{de_\phi}{dt} = -\dot{\phi}e_r \sin\theta - \dot{\phi}e_\theta \cos\phi$ [4]

(b) Show that

(i) $e_r \bullet e_\theta = 0$ and that [2]

(ii) $e_r \bullet e_r = 1$ [2]

(c) A particle describes a path with position vector
$\bar{r} = a\cos wt\hat{i} + b\sin wt\hat{j}$, show that

(i) The path is an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ [3]

(ii) The acceleration is directed towards the origin [2]

(iii) $\int_t^{t+t_1} \bar{r} X d\bar{r} = wabt_1\hat{k}$ and interpret it. [3]

8. A given rigid body R, will in general have six degrees of freedom and its position at any time t relative to a given inertial frame S, can be found using two equations of motion.

$$\frac{d\vec{P}}{dt} = \vec{F}\ , \quad \frac{d\vec{L}}{dt} = \vec{M}$$

Where $\vec{F}$ is the total external force acting on the body and $\vec{M}$ is the total moment of the external forces.

(a) Write down in integral form the expressions for the total linear momentum $\vec{P}$ and the total angular momentum $\vec{L}$. [2]

(b) Show that if the rigid body rotates with angular velocity ϖ, then the kinetic energy of rotation is given by

$$T = \frac{1}{2}\mathrm{I}\varpi^2$$ [14]

(c) A ladder of length l and weight W_l has one end against a vertical wall which is frictionless and the other end on the ground assumed horizontal.

Prove that a man of weight W_m will be able to climb the ladder without having it slip if the coefficient of friction μ between the ladder and the ground is at least

$$\frac{W_m + \frac{1}{2}W_l}{W_m + W_l}\cot\alpha \qquad [8]$$

9. (a) A particle of mass m moves in a conservative force field.

Find

(i) The Lagrangian function, [3]

(ii) The equations of motion in cylindrical coordinates (ρ, ϕ, z). [6]

(b) Suppose that the forces acting on a system of particles are derived from a potential function V, that is, suppose that the system is conservative. Prove that if $L = T - V$ is Lagrangian function then $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_\alpha}\right] - \frac{\partial L}{\partial q_\alpha} = 0$ [5]

(c) (i) Set up the Lagrangian of a simple pendulum and [5]

(ii) Obtain an equation describing its motion. [5]

10. (a) What do you understand by the term frame of reference? [2]

(b) For the position vector $\bar{r}$, the time derivative in an inertial frame (fixed frame, F) is obtained by adding the term $\bar{\omega} X \bar{r}$ to the time derivative in a rotating frame(moving frame, M), namely

$$\left[\frac{d\bar{r}}{dt}\right]_F = \left[\frac{d\bar{r}}{dt}\right]_M + \bar{\omega} X \bar{r}$$

where $\bar{\omega}$ is the angular velocity of the rotating frame. Find the relationship between the acceleration vectors in the two frames. [6]

(c) An xyz frame of reference is rotating with respect to an XYZ frame of reference, having the same origin and assumed to be fixed in space (i.e. it is an inertial frame). The angular velocity of the xyz relative to the XYZ system is given by

$$\vec{\omega} = 2t\hat{i} - t^2\hat{j} + (2t+4)\hat{k}$$

where t is the time. The position vector of a particle at time t as observed in the xyz system is given by

$$\vec{r} = (t^2+1)\hat{i} - 6t\hat{j} + 4t^3\hat{k}$$

(i) Find the apparent velocity and the true velocity at time $t = 1$ in the above problem. [3]

(ii) Find the apparent acceleration and the true acceleration of the particle in the above problem (i.e. when $t = 1$) [10]

(iii) Referring to the above information, find the coriolis acceleration, the centripental acceleration and their magnitudes at $t = 1$. [3]

11. (a) (i) State the Hamilton's principle. [2]

(ii) Define the Hamiltonian, H, in terms of the Lagrangian L. [2]

(b) If the Hamiltonian $H = \sum p_\alpha \dot{q}_\alpha - L$, where the summation extends from $\alpha = 1$ to n, is expressed as a function of coordinates q_α and the momenta p_α, prove Hamilton's equations $\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$, $\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$

regardless of whether H

(i) does not contain t explicitly [10]

(ii) does contain t explicitly [10]

END OF QUESTION PAPER