NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 2103

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA2103: THEORETICAL MECHANICS

JULY 2005 SUPPLEMENTARY EXAMINATIONS

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Sections B.

SECTION A: Answer ALL questions in this section [28].

A1. Show that the velocity of a particle is independent of the origin to which its position vector is referred. [4]

A2. Determine the number of degrees of freedom in each of the following cases:

(a) A rigid body moving parallel to a fixed plane. [2]

(b) A system consisting of a thin rigid rod which can move freely in space and a particle which is constrained to move on the rod. [2]

A3. A projectile is launched at an angle α from a cliff of height H above sea level. If it falls into the sea at a distance D from the base of the cliff, prove that its maximum height above sea level is

$$H = \frac{D^2 \tan^2 \alpha}{4(H + D \tan \alpha)}.$$

[5]

A4. A particle moves along a curve whose parametric equations are:

$$x = 3e^{-2t}, y = 4\sin 3t, z = 5\cos 3t$$

where t is the time.
Find the velocity and acceleration at any time t. [3]

A5. Given that the position vector of a particle in terms of plane polar coordinates is given by:

$$\vec{r} = re_r,$$

where $e_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ and
$e_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$
Show that the velocity and acceleration of the particle are given by:

(a) $\vec{V} = \dot{r}e_r + r\dot{\theta}e_\theta$. [2]

(b) $\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)e_r + \frac{e_\theta}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$. [2]

A6. A ladder of length l and weight W_l has one end against a vertical wall which is frictionless and the other end on the ground assumed horizontal.

Prove that a man of weight W_l will be able to climb the ladder without having it slip if the coefficient of friction μ between the ladder and the ground is at least

$$\frac{W_m + \frac{1}{2}W_l}{W_m + W_l}\cot\alpha$$

[8]

SECTION B: Answer THREE questions in this section [72].

B7. (a) State Hamilton's principle. [2]

(b) Define the Hamiltonian, H, in terms of the Lagrangian L. [2]

(c) If the Hamiltonian, $H = \sum p_\alpha \dot{q}_\alpha - L$, where the summation extends from $\alpha = 1$ to n, is expressed as a function of coordinates q_α and the momenta p_α, prove Hamilton's equations

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}, \qquad \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$$

regardless of whether H:

(i) does contain t explictly, [10]

(ii) does not contain t explictly. [10]

B8. (a) A particle of mass m moves in a conservative force field. Find

(i) The Lagrangian function, [3]

(ii) The equations of motion in cylindrical coordinates (ρ, ϕ, z). [6]

(b) Suppose that the forces acting on a system of particles are derived from a potential function V, that is, suppose that the system is conservative. Prove that if $L = T - V$ is the Lagrangian function then $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_\alpha}\right] - \frac{\partial L}{\partial q_\alpha} = 0.$ [5]

(c) (i) Set up the Lagrangian of a simple pendulum and [5]

(ii) Obtain an equation describing its motion. [5]

B9. Given that in spherical coordinates the position vector of a particle is given by $\vec{r} = re_r$, Where

$e_r = \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta$
$e_\theta = \mathbf{i}\cos\theta\cos\phi + \mathbf{j}\cos\theta\sin\phi - \mathbf{k}\sin\theta$
$e_\phi = -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi$

(a) Show that

(i) $\frac{de_r}{dt} = \dot{\theta}e_\theta + \dot{\phi}e_\phi\sin\theta.$ [4]

(ii) $\frac{de_\theta}{dt} = -\dot{\theta}e_r + \dot{\phi}e_\phi\cos\theta.$ [4]

(iii) $\frac{de_\phi}{dt} = -\dot{\phi}e_r\sin\theta - \dot{\phi} - \dot{\phi}e_\theta\cos\theta.$ [4]

(b) Show that

(i) $e_r \cdot e_\theta = 0$ and that [2]

(ii) $e_r \cdot e_r = 1.$ [2]

(c) A particle describes a path with position vector $\vec{r} = a\cos\omega t\mathbf{i} + b\sin\omega t\mathbf{j}$, show that

(i) The path is an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ [3]

(ii) The acceleration is directed towards the origin. [2]

(iii) $\int_t^{t+t_1} \vec{r} \times d\vec{r} = \omega abt_1\mathbf{k}$ and interpret it. [3]

B10. A given rigid body R, will in general have six degrees of freedom and its position at any time t relative to a given inertial frame S, can be found using the two equations of motion

$$\frac{d\vec{P}}{dt} = \vec{F}, \qquad \frac{d\vec{L}}{dt} = \vec{M},$$

Where $\vec{F}$ is the total external force acting on the body and $\vec{M}$ is the total moment of the external forces.

(a) Write down in integral form the expressions for the total linear momentum $\vec{P}$ and the total angular momentum $\vec{L}$. [2]

(b) Show that the angular momentum $\vec{L}$ of the rigid body R relative to an origin fixed in the rigid body can be written in terms of the angular velocity ϖ to become

$$\vec{L} = I\varpi$$

[15]

(c) Show also that if the rigid body rotates with angular velocity ϖ, then the kinetic energy of rotation is given by

$$T = \frac{1}{2}I\varpi^2$$

where I is the moment of inertia about the axis. [7]

END OF QUESTION PAPER