NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 2103

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA2103: THEORETICAL MECHANICS

DECEMBER 2005 EXAMINATIONS

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Sections B.

SECTION A: Answer ALL questions in this section [28].

A1. Show that if S is an inertial frame, then a frame of reference S', whose origin O' is moving with uniform velocity U with respect to S, is also an inertial frame. [4]

A2. The position of a particle in terms of plane polar coordinates is given by $\vec{r} = re_r$. Show that the velocity and acceleration of the particle are given by:

(a) $\vec{V} = \dot{r}e_r + r\dot{\theta}e_\theta$. [1]

(b) $\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right) e_r + \frac{e_\theta}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$. [3]

A3. Find the range of a projectile which is fired with speed V at angle α to an inclined plane which is itself inclined at an angle β to the horizontal. For fixed V, what is the maximum range? [6]

A4. A particle of mass 2 moves in a force field depending on time given by $\vec{F} = 24t^2\mathbf{i} + (36t - 16)\mathbf{j} - 12t\mathbf{k}$. Given that at $t = 0$ the particle is located at $\vec{r}_0 = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and has velocity $\vec{V}_0 = 6\mathbf{i} + 15\mathbf{j} - 8\mathbf{k}$ find:

(a) the velocity, [2]

(b) the position vector at any time t. [2]

A5. An xyz frame of reference is rotating with respect to XYZ frame of reference having the same origin and assumed to be fixed in space. The angular velocity of the xyz relative to the XYZ system is given by

$$\vec{\omega} = 2t\mathbf{i} - t^2\mathbf{j} + (2t+4)\mathbf{k}$$

where t is the time. The position vector of a particle at time t as observed in the xyz system is given by

$$\vec{r} = (t^2+1)\mathbf{i} - 6t\mathbf{j} + 4t^3\mathbf{k}.$$

Find

(a) the apparent velocity and the true velocity of the particle at time $t = 1$, [4]

(b) the apparent acceleration and the true acceleration of the particle at time $t = 1$. [6]

SECTION B: Answer THREE questions in this section [72].

B6. A particle is attracted to a fixed point O by a force $F(r)$ per unit mass, where r is the distance of the particle from O. Assuming that the particle moves in a plane through O, prove that

$$\frac{d^2u}{d\theta^2} + u = \frac{F(u^{-1})}{h^2u^2}$$

where h is a constant, $u = r^{-1}$, and θ is the angle the line from O to the particle makes with some fixed direction in the plane.

Now suppose that

$$F(r) = k\left[4\left(\frac{a}{r}\right)^2 - 3\left(\frac{a}{r}\right)^3\right]$$

and the particle is projected from a point at a distance a from O with a component of velocity $\sqrt{ak}$ along the line from O to the particle extended and an equal component perpendicular to the line.

(a) Show that the maximum and minimum distances of the particle from O are $2a$ and $\frac{2}{3}a$ respectively.

(b) Find the angle turned through by the radius vector between the directions corresponding to the first maximum and subsequent minimum. [24]

B7. (a) In terms of the Hamiltonian $H(p_\alpha, q_\alpha, t)$ give the equations of motion of a system (sometimes called Hamilton's canonical equations). [2]

(b) For conservative systems, express the Hamiltonian H in terms of the total energy (kinetic (T) and potential (V))of the system. [1]

(c) If the Hamiltonian, $H = \sum p_\alpha \dot{q}_\alpha - L$, where the summation extends from $\alpha = 1$ to n, is expressed as a function of coordinates q_α and the momenta p_α, prove Hamilton's equations

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}, \qquad \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$$

where the Hamiltonian H contains t explicitly. [10]

(d) If the Hamiltonian H is independent of t explicitly, prove that

(i) it is a constant, [4]

(ii) it is equal to the total energy of the system. [7]

B8. (a) A bead slides without friction on a frictionless wire in the shape of a cycloid with equations

$$x = a(\theta - \sin\theta), \quad y = a(1 + \cos\theta)$$

where $0 \leq \theta \leq 2\pi$.
Find

(i) The Lagrangian function, [5]

(ii) The equations of motion . [5]

(b) A particle of mass m moves in a conservative force field.
Find

(i) the Langrangian function, [3]

(ii) the equation of motion in cylindrical coordinates. [6]

(c) Suppose that the forces acting on a system of particles are derived from a potential function V, that is, suppose that the system is conservative. Prove that if $L = T - V$ is the Lagrangian function then $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_\alpha}\right] - \frac{\partial L}{\partial q_\alpha} = 0$. [5]

B9. (a) State the following theorems:

(i) Euler's theorem with reference to rotation of a rigid body about a fixed point, [1]

(ii) Chasle's theorem with reference to the general motion of a rigid body. [1]

(b) Define the following:

(i) moment of a forcce, [1]

(ii) moment of inertia. [1]

(c) A ladder of length l and weight W_l has one end against a vertical wall which is frictionless and the other end on the ground assumed horizontal.

Prove that Dr.Woong of weight W_l will be able to climb the ladder without having it slip if the coefficient of friction μ between the ladder and the ground is at least

$$\frac{W_m + \frac{1}{2}W_l}{W_m + W_l}\cot\alpha$$

[10]

(d) Two particles of mass m_1 and m_2 respectively are connected by a rigid massless rod of length a and move freely in a plane. Show that the moment of inertia of the system about an axis perpendicular to the plane and passing through the centre of mass is μa^2 where the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

[10]

END OF QUESTION PAPER