NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

December 2002 EXAMINATION

SMA2104 PARTIAL DIFFERENTIAL EQUATIONS

3 Hours

Answer ALL questions in Section A and any FOUR questions in Section B.

SECTION A : Answer ALL questions from this section.

[28 Marks]

1. Find the Fourier series of the function

$$f(x) = x, \qquad -\pi < x \leq \pi$$
$$f(x + 2\pi) = f(x).$$

[6 Marks]

2. Use the method of separation of variables to solve the partial differential equation

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y},$$

given that

$$u(0, y) = 2e^{-3y} + 3.$$

[4 Marks]

3. Find the Fourier cosine transforms of the functions

(a) $H(t-1) - H(t)$, where H is the unit step function,

[3 Marks]

(b) $\delta(t-1)\cos t$, where δ is the unit impulse function.

[3 Marks]

4. Use Laplace transforms to solve the initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 1 + e^{-2t}, \quad y(0) = 1,\ y'(0) = -1.$$

[6 Marks]

5. Find the general solution of the partial differential equation

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{1}{y}\frac{\partial u}{\partial x} = x \quad (y \neq 0).$$

[6 Marks]

SECTION B - Answer any FOUR questions from this section.

[72 Marks]

6. (a) Show that, for an odd function $f(x)$ of period 2ℓ which satisfies Dirichlet's conditions,

$$E_n = \int_{-\ell}^{\ell} (f(x) - S_n(x))^2 \, dx$$

is minimised if

$$b_r = \frac{2}{\ell}\int_0^{\ell} f(x)\sin(\frac{r\pi x}{\ell})dx \qquad (\ r = 0, 1, 2, 3...),$$

where

$$S_n(x) = \sum_{r=1}^{n} b_r \sin(\frac{r\pi x}{\ell}).$$

[6 Marks]

(b) Find the Fourier half range sine series for the function defined by

$$f(x) = x - 1\ ;\ 0 < x < 2.$$

[6 Marks]

i. Sketch the function represented by the series over the interval $-4 \leq x \leq 4$,

[3 Marks]

ii. Find a series representation for π.

[3 Marks]

7. (a) Find the Fourier transforms of

   i. $H(x+2) - H(x-2)$, where H is the unit step function,

   [3 Marks]

   ii. $(H(x+2) - H(x-2)) \cos x$.

   [4 Marks]

   iii. $H(x-1) - H(x)$.

   [3 Marks]

   (b) i. Use Parseval's theorem on $H(x+2) - H(x-2)$ to find an expression for π.

   [3 Marks]

   ii. The convolution theorem states that, if

$$g(x) = \int_{-\infty}^{\infty} f_1(t) f_2(x-t)\, dt,$$

   then

$$G(\omega) = F_1(\omega) F_2(\omega).$$

   Hence find the function which has as its Fourier transform the product of the transforms of $H(x+2) - H(x-2)$ and $H(x-1) - H(x)$.

   [5 Marks]

8. Let $F(s)$ denote the Laplace Transform of $f(t)$, i.e.

$$F(s) = \int_0^{\infty} f(t) e^{-st}\, dt.$$

   (a) Use the definition given above to find the Laplace transform of $\cosh t$.

   [3 Marks]

   (b) Show that the Laplace transform of $t f(t)$ is $-\dfrac{dF}{ds}$. Hence find the Laplace transform of $t \cosh t$.

   [4 Marks]

   (c) Show that the Laplace transform of $H(t-k) f(t-k)$ is $e^{-ks} F(s)$. Hence find the Laplace transform of $H(t-1)(t-1)\cosh(t-1)$.

   [4 Marks]

   (d) Use Laplace transforms to solve the boundary value problem

$$\frac{d^2y}{dt^2} + y = H(t-1)(t-1)\cosh(t-1), \; y(0) = 1, y'(0) = -1.$$

   [7 Marks]

9. Consider the boundary value problem

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \lambda y = 0, \quad y(0) = y(1) = 0.$$

(a) Write the differential equation in Sturm-Liouville form, i.e.

$$\frac{d(r\frac{dy}{dx})}{dx} + (q + \lambda p)y = 0.$$

[3 Marks]

(b) Find the eigenvalues, λ_n, and the corresponding eigenfunctions, $f_n(x)$, of the boundary value problem.

[6 Marks]

(c) Verify that the eigenfunctions satisfy the orthogonality conditions for a regular Sturm-Liouville problem

$$\int_a^b p(x) f_n(x) f_m(x)\, dx = 0.$$

[3 Marks]

(d) Given that the coefficients in the generalised Fourier series for the function

$$g(x) = x(1 - x), \quad g(0) = g(1) = 0$$

are given by

$$c_n = \frac{\int_a^b p(x) g(x) f_n(x)\, dx}{\int_a^b p(x)[f_n(x)]^2\, dx},$$

find the generalised Fourier series for $g(x)$ in terms of these eigenfunctions.

[6 Marks]

10. (a) The Fourier Sine transform of a function $f(x)$ is given by

$$F_s(\omega) = \int_0^\infty f(x) \sin(\omega x)\, dx.$$

Stating clearly any assumptions you make, find the Fourier Sine transform of $\frac{d^2y}{dx^2}$ in terms of the transform of y.

[5 Marks]

(b) Use a Fourier sine transform to solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (0 < y < 1, \; 0 < x < \infty),$$

$$u(0, y) = y(1 - y),$$

$$u(x, 0) = u(x, 1) = 0.$$

[13 Marks]

11. A square plate, of side 1 metre, is insulated on each face. Three edges are held at a temperature of 25 degrees Celcius, the other edge is held at 100 degrees Celcius.

(a) Show how this situation can be modelled by the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (0 < x < a, 0 < y < b)$$

given that

$$u(0, y) = u(a, y) = 0 \qquad (0 < y < b)$$

and

$$u(x, 0) = 0 \quad u(x, b) = A \quad (0 < x < a).$$

[3 Marks]

(b) Use the method of separation of variables to solve this boundary value problem. Hence find the steady state temperature of the centre of the plate, correct to the nearest integer.

[10 Marks]

(c) Find the steady state temperature of the centre of the plate if two adjacent edges of the plate are held at 25 degrees Celcius, the other two edges being held at 100 degrees Celcius.

[5 Marks]

END OF EXAMINATION PAPER