NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

December 2004 EXAMINATION

SMA2104 PARTIAL DIFFERENTIAL EQUATIONS

3 Hours

Answer ALL questions in Section A and any FOUR questions in Section B.

SECTION A : Answer ALL questions from this section.

[28 Marks]

1. Find the Fourier series of the function

$$f(x) = x + \pi, \qquad -\pi < x \leq \pi$$

$$f(x + 2\pi) = f(x).$$

[7 Marks]

2. Use the method of separation of variables to solve the partial differential equation

$$\frac{\partial u}{\partial x} = -2\frac{\partial u}{\partial y},$$

given that

$$u(0, y) = 3e^{-2y} + 4.$$

[4 Marks]

3. Find the real part of the Fourier transform of the function $H(x-1)e^{-2x}$, where H is the unit step function.

[5 Marks]

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4. Use Laplace transforms to solve the initial value problem

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 2 - 6e^{-3t}, \quad y(0) = 1, \ y'(0) = 4.$$

[7 Marks]

5. Find the general solution of the partial differential equation

$$y\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} = x.$$

[5 Marks]

SECTION B - Answer any FOUR questions from this section.

[72 Marks]

6. (a) Show that, for an odd function $f(x)$ of period 2ℓ which satisfies Dirichlet's conditions,

$$E_n = \int_{-\ell}^{\ell} (f(x) - S_n(x))^2 \, dx$$

is minimised if

$$b_r = \frac{2}{\ell}\int_0^{\ell} f(x)\sin(\frac{r\pi x}{\ell})dx \qquad (r = 0, 1, 2, 3...),$$

where

$$S_n(x) = \sum_{r=1}^{n} b_r \sin(\frac{r\pi x}{\ell}).$$

[6 Marks]

(b) Find the Fourier half range sine series for the function defined by

$$f(x) = 4 \ ; \ 0 < x < 2.$$

[6 Marks]

i. Sketch the function represented by each of the series over the interval $-4 \leq x \leq 4$.

[3 Marks]

ii. Find a series representation for π.

[3 Marks]

7. (a) Find the Fourier transform of $H(x+1)-H(x-1)$, where H is the unit step function.

[4 Marks]

(b) Show that the Fourier transform of $f(x+a)$ is $e^{i\omega a}F(\omega)$, where $F(\omega)$ is the Fourier transform of $f(x)$. Hence write down the Fourier transform of $H(x+2)-H(x)$.

[5 Marks]

(c) Use Parseval's theorem on $H(x+2)-H(x)$ to find an expression for π.

[3 Marks]

(d) The convolution theorem states that, if

$$g(x) = \int_{-\infty}^{\infty} f_1(t)f_2(x-t)\, dt,$$

then

$$G(\omega) = F_1(\omega)F_2(\omega).$$

Hence find the function which has as its Fourier transform the product of the transforms of $H(x+2)-H(x)$ and $H(x+1)-H(x-1)$.

[6 Marks]

8. Let $F(s)$ denote the Laplace Transform of $f(t)$, i.e.

$$F(s) = \int_{0}^{\infty} f(t)e^{-st}\, dt.$$

(a) Use the definition given above to show that the Laplace transform of $\sinh t$ is $\frac{1}{s^2-1}$.

[3 Marks]

(b) Show that the Laplace transform of $tf(t)$ is $-\frac{dF}{ds}$. Hence find the Laplace transform of $t \sinh t$.

[4 Marks]

(c) Show that the Laplace transform of $H(t-k)f(t-k)$ is $e^{-ks}F(s)$. Hence find the Laplace transform of $H(t-2)(t-2)\sinh(t-2)$.

[4 Marks]

(d) Use Laplace transforms to solve the boundary value problem

$$\frac{d^2y}{dt^2} + y = H(t-2)(t-2)\sinh(t-2), \; y(0) = 1, y'(0) = -1.$$

[7 Marks]

9. Consider the boundary value problem

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + \lambda y = 0, \qquad y(0) = y(\pi) = 0.$$

(a) Write the differential equation in Sturm-Liouville form, i.e.

$$\frac{d(r\frac{dy}{dx})}{dx} + (q + \lambda p)y = 0.$$

[3 Marks]

(b) Find the eigenvalues, λ_n, and the corresponding eigenfunctions, $f_n(x)$, of the boundary value problem.

[8 Marks]

(c) Verify that the eigenfunctions satisfy the orthogonality conditions for a regular Sturm-Liouville problem

$$\int_a^b p(x)f_n(x)f_m(x)\ dx = 0, \quad n \neq m.$$

[3 Marks]

(d) Given that the coefficients in the generalised Fourier series for the function

$$g(x) = e^{-2x}, \quad (0 < x < \pi), \quad g(0) = g(\pi) = 0$$

are given by

$$c_n = \frac{\int_a^b p(x)g(x)f_n(x)\ dx}{\int_a^b p(x)[f_n(x)]^2\ dx},$$

find the generalised Fourier series for $g(x)$ in terms of these eigenfunctions.

[4 Marks]

10. The Fourier Sine transform of a function $f(x)$ is given by

$$F_s(\omega) = \int_0^\infty f(x)\sin(\omega x)\ dx$$

and the Fourier Sine integral is given by

$$\frac{2}{\pi}\int_0^\infty F_s(\omega)\sin(\omega x)\ d\omega.$$

(a) Stating clearly any assumptions you make, find the Fourier Sine transform of $\frac{d^2y}{dx^2}$ in terms of the transform of y and initial conditions.

[5 Marks]

(b) Use a Fourier sine transform to solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (0 < y < 1, \;\; 0 < x < \infty),$$

$$u(0, y) = 1,$$

$$u(x, 0) = u(x, 1) = 0.$$

[13 Marks]

11. A square plate, of side 1 metre, is insulated on each face. Three edges are held at a temperature of 0 degrees Celcius, the other edge is held at 100 degrees Celcius.

(a) Show how this situation can be modelled by the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (0 < x < a, 0 < y < b)$$

given that

$$u(0, y) = u(a, y) = 0 \quad (0 < y < b)$$

and

$$u(x, 0) = 0 \quad u(x, b) = A \quad (0 < x < a).$$

[3 Marks]

(b) Use the method of separation of variables to solve this boundary value problem. Hence find the steady state temperature of the centre of the plate, correct to the nearest integer.

[12 Marks]

(c) Find the steady state temperature of the centre of the plate if two adjacent edges of the plate are held at 0 degrees Celcius, the other two edges being held at 100 degrees Celcius.

[3 Marks]

END OF EXAMINATION PAPER