NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 2104

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA2104: PARTIAL DIFFERENTIAL EQUATIONS

DECEMBER 2005

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY TWO** questions from Sections B.

SECTION A: Answer ALL questions in this section [40].

A1. (a) Classify the following equation

$$yu_{xx} + u_{yy} = 0.$$

[3]

(b) The general temperature distribution for an insulated bar of length L, density ρ, specific heat c_p, and thermal conductivity k is governed by

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2} \quad , \quad 0 \leq x \leq L$$

in which x is position, T is temperature, and t is time. Show that the equation can be reduced to the form

$$u_\tau = u_{\xi\xi} \quad , \quad for \quad 0 \leq x \leq 1$$

by using the change of variable

$$\xi = \frac{x}{L} \quad , \quad \tau = \frac{kt}{\rho c_p L^2} \quad , \quad u = \frac{T}{T_{ref}}$$

where T_{ref} is a reference temperature. [7]

A2 using the method of characteristics, solve the initial value problem

$$xu_x + 2yu_y + u_z = 3u$$

with $u = \psi(x, y)$ on the plane $z = 0$. [5]

A3. (a) State the uniform convergence Theorem. [2]

(b) Find the Fourier series of

$$f(x) = x^2 \quad , \quad x \in [-\pi, \pi].$$

Hence, by using Parseval's Theorem, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

[5,5]

(c) Show that the Fourier series of

$$f(x) = x^2$$

converges uniformly on $-\pi < x < \pi$. [3]

A4. (a) Find the complex Fourier series of

$$f(x) = e^x \quad , \quad x \in [-\pi, \pi].$$

[5]

(b) Find the Fourier transform of

$$f(x) = e^{-|x|}.$$

[5]

SECTION B: Answer TWO questions in this section [60].

B5. (a) Classify each of the following equations:

(i)

$$y^2 u_{xx} - x^2 u_{yy} = 0 \quad , \quad x > 0, \quad y > 0,$$

[1]

(ii)

$$u_{xx} - 2xu_{xy} + x^2 u_{yy} = 0,$$

[1]

(iii) [illegible]

$$x^2u_{xx} + y^2u_{yy} = 0 \quad , \quad x > 0, y > 0.$$

[1]

(b) Transform the hyperbolic equation

$$y^2u_{xx} - x^2u_{yy} = 0$$

into canonical form. [6]

(c) Consider the partial differential equation

$$yu_{xx} + (x + y)u_{xy} + xu_{yy} = 0$$

(i) show that the equation is hyperbolic in the region where $x \neq y$, [2]

(ii) show that the characteristics curves are given by

$$\xi(x, y) = y - x \quad , \quad \eta(x, y) = y^2 - x^2,$$

[4]

(iii) in the region where the equation is hyperbolic show that the canonical form is given by

$$\xi u_{\xi\eta} + u_\eta = 0,$$

(Note: $y = \frac{1}{2}\left(\frac{\eta}{\xi} + \xi\right), \ x = \frac{1}{2}\left(\frac{\eta}{\xi} - \xi\right)$) [9]

(iv) hence, show that the general solution of the above problem is

$$u(x, y) = \frac{1}{y - x}\int f(\eta)\, d\eta + g(y - x)$$

where $\eta = y^2 - x^2$. [6]

B6. Consider the one-dimensional wave equation

$$u_{tt} - c^2u_{xx} = 0 \quad , \quad -\infty < x < \infty, \quad t > 0$$

and the homogeneous initial conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad -\infty < x < \infty.$$

(a) By using method of characteristics, show that the solution of the wave equation is given by

$$u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\mu)d\mu.$$

Hence, solve

$$u_{tt} - c^2u_{xx} = 0 \quad , \quad -\infty < x < \infty \quad , \quad t > 0$$

subject to

$$u(x, 0) = e^{-x^2} \quad , \quad u_t(x, 0) = 0.$$

[10,5]

(b) Consider the wave equation in a semi-infinity domain $0 < x < \infty$, that is

$$u_{tt} - c^2 u_{xx} = 0 \quad , \quad 0 < x < \infty \quad , \quad t > 0$$

$$I.C : u(x,0) = f(x) \quad , \quad u_t(x,0) = g(x) \quad , \quad 0 < x < \infty$$

$$B.C : u(0,t) = 0$$

and the compatibility condition

$$f(0) = g(0) = 0.$$

Show that the solution of the wave equation in this case is

$$u(x,t) = \begin{cases} \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\mu)d\mu & \text{if } x \geq ct \\ \frac{1}{2}[f(x+ct) - f(x-ct)] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\mu)d\mu & \text{if } x < ct \end{cases}.$$

[15]

B7. (a) The double Fourier Sine Series of a function of two variables $f(x,y)$ is given by

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2}$$

where

$$A_{nm} = \frac{4}{L_1 L_2}\int_0^{L_1}\int_0^{L_2} f(x,y) \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2}\, dxdy, \quad 0 < x < L_1, \quad 0 < y < L_2.$$

Find the double Fourier series of

$$f(x,y) = xy.$$

[10]

(b) Solve the following Laplace equation by using the method of separation of variables

$$u_{tt} = c^2(u_{xx} + u_{yy}) \quad , \quad 0 < x < 1 \quad , \quad 0 < y < 2$$

subject to the boundary conditions

$$u(0,y,t) = u(1,y,t) = 0$$

$$u(x,0,t) = u(x,2,t) = 0$$

and the initial conditions

$$u(x,y,0) = xy$$

$$u_t(x,y,0) = 0.$$

[20]

B8. (a) The Fourier cosine transform of the function $f(x)$ is defined by

$$\mathcal{F}_c\{f(x)\} = \frac{2}{\pi}\int_0^\infty f(x)\cos(\mu x)d\mu$$

Find the cosine transform for

$$f(x) = e^{-x}\cos x.$$

[12]

(b) The temperature $u(x,t)$ in the semi-infinite rod, $0 < x < \infty$, is determined by the differential equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

and the conditions

$$u(x,0) = 0 \quad , \quad u_x(0,t) = -\theta_0$$

where θ_0 is a constant.
Making use of the cosine transforms, show that

$$u(x,t) = \frac{2\theta_0}{\pi}\int_0^\infty \frac{1 - \exp^{-k\mu^2 t}}{\mu^2}\cos(\mu x)d\mu.$$

[18]

END OF QUESTION PAPER