NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY
SMA 2106

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA 2106: PROBABILITY THEORY

JANUARY 2004

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B

SECTION A: [25 marks]

Answer **ALL** questions from this section.

A1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and A be a subset of Ω. The collection $\mathcal{G}$ of all subsets of Ω of the form $G = A \cap F$, $F \in \mathcal{F}$, is called the trace of $\mathcal{F}$ on A.

(a) Show that the trace of $\mathcal{F}$ on A is a $\sigma - algebra$ in A [6]

(b) If $A \in \mathcal{F}$ and $P(A) > 0$, define a set function Q_A on $\mathcal{G}$ by $Q_A(G) = \dfrac{P(G)}{P(A)}$.
Show that Q_A is a probability measure on $\mathcal{G}$. [6]

A2. Find the value of k for which the following function is a joint density function of X and Y:

$$f_{X,Y}(x,y) = \begin{cases} k(6-x-y) & \text{if } (x,y) \in \{ (x,y) \in \mathbb{R}^2 : x \geq 0, x+y \leq 6, x+2y \geq 4\} \\ 0 & \text{otherwise} \end{cases}$$ [6]

A3. An experiment consists of throwing two unbiased dice and let S be the score, i.e the sum of the numbers that appear on both dice and let π be the product of them. Define the following random variables X and Y as follows:

$$X = \begin{cases} 1 & \text{if } S \text{ is even} \\ 0 & \text{if } S \text{ is odd} \end{cases}$$

$$Y = \begin{cases} 1 & \text{if } \pi \text{ is even} \\ 0 & \text{if } \pi \text{ is odd} \end{cases}$$

(a) Describe the distribution of the random vector (X,Y). [4]

(b) Calculate the distribution function $F_{X,Y}(x,y)$. [3]

SECTION B: [75 marks]

Answer **ANY THREE** questions from this section. Each question carries 25 marks.

B4. (a) The joint probability density function of (X,Y) is given by

$$f_{X,Y}(x,y) = \begin{cases} c(xy+y^2) \text{ if } 0 \le x \le 2;\ 0 \le y \le 2 \\ 0 \text{ otherwise} \end{cases}$$

(i) Find c [2]

(ii) Calculate $P(X+Y<2)$ [5]

(iii) Calculate the coefficient of correlation ρ_{XY}. [6]

(b) Two points A and B are randomly selected on the circumference of a circle, centre origin O and radius r. Calculate the expectation of the area of triangle AOB. [8]

(c) A random vector (X,Y) has the following characteristics:
E[X] = -1; E[Y] = 1; Var[X] = 4 and Var[Y] = 9 and the coefficient of correlation is $\rho_{XY} = \frac{1}{2}$.
Calculate $E(Z)$ where $Z = (X-Y)^2$. [4]

B5. (a) A random vector (X,Y) is normally distributed with parameters (μ_X, μ_Y); σ_X; σ_Y and ρ_{XY}.

(i) Calculate the conditional density function of Y given X, $f_{Y|X}(x,y)$. [8]

(ii) Hence or otherwise, calculate the conditional expectation $E[Y|X=x]$. [3]

(b) The random vector (X,Y) has the following characteristics:

E[X] = -2; E[Y] = 3 and the covariance matrix is $K = \begin{pmatrix} 16 & 12 \\ 12 & 25 \end{pmatrix}$

Calculate the joint density function $f_{X,Y}(x,y)$. [6]

(c) The joint probability density function of a random vector (X,Y) is given by
$f_{X,Y}(x,y) = \frac{1}{\pi} e^{-\frac{1}{2}(x^2+2xy+5y^2)}$.
Calculate $f_X(x)$ and ρ_{XY}. [5,3]

B6. (a) A random variable X is normally distributed with mean μ and variance σ^2.
Calculate the characteristic function of X. [6]
Hence or otherwise calculate E[X] and Var[X] if they exist. [3,3]

(b) Given that the characteristic function of a continuous random variable X is

$$\Theta_X(t) = \frac{e^{itm}}{1+\frac{\sigma^2 t^2}{2}}$$

Find the density function of X and calculate E(X) and Var(X) as functions of m and σ. [7,3,3]

B7. (a) A sequence $X_1, X_2, \ldots,$ of random variables satisfies $E[X_k^2] < \alpha^2,\ \forall k = 1, 2, \ldots.$ Given also that X_k and X_{k+1} are independent

(i) Find the upper bound of the probability
$$P\left(\left|\frac{X_k + X_{k=1}}{2} - \frac{E(X_k) + E(X_{k+1})}{2}\right| \geq \sigma_k + \sigma_{k+1}\right)$$
where $\sigma_j = \sqrt{Var(X_j)}, \forall j = 1, 2, \ldots.$ [7]

(ii) Prove that the sequence satisfies Markov's law of large numbers. [6]

(b) A random variable X has a Poisson distribution with parameter λ. Prove that when $\lambda \to \infty$, then
$Z = \dfrac{X - \lambda}{\sqrt{\lambda}}$ has a standard normal distribution. [6]

(c) Let f be an increasing function and assume that $E[f(x)]$ exists, where X ia a nonnegative random variable.
Prove that $\forall \varepsilon, P(X \geq \varepsilon) \leq \dfrac{E[f(x)]}{f(\varepsilon)}$. [6]

B8. (a) Calculate the moment generating function of a random variable X whose probability density function is
$f_Y(y) = \dfrac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{y-a}{b})^2}, \quad -\infty < y < \infty.$ [7]

(b) Calculate $E(X)$ and $Var(X)$ if X is a random variable with density
$$f_X(x) = \begin{cases} \theta e^{-\theta x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$ [3,4]

(c) A random vector (X,Y) has joint density function
$$f_{X,Y}(x, y) = \begin{cases} e^{-(x+y)}, & \text{if, } x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(i) Calculate E(XY). [6]

(ii) Determine whether or not X and Y are independent. [5]

END OF QUESTION PAPER