NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 2106

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA 2106: PROBABILITY THEORY

DECEMBER 2004

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B

SECTION A: [40 marks]

Answer **ALL** questions from this section.

A1. (a) Suppose that A, B and C are events defined on a probability space $(\Omega, \mathcal{F}, P)$. Prove that

(i) P(A + B + C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC) [4]

(ii) If $C \supset AB$, then $P(AB + AC + BC) \leq P(A + B)$ [4]

(b) Suppose that $\{H_k\}$; (k = 2, 3, ...,14) is a disjoint family of events and A is another event such that $P(A|H_k) = \frac{14-k}{51}$ and $P(H_k) = \frac{1}{13}$

(i) Calculate P(A) [5]

(ii) Give an expression for $P(H_k|A)$ [3]

A2. Two dice are thrown and the random variables X and Y are defined as follows

$$X = \begin{cases} 1 & \text{if the sum of the numbers showing is even} \\ 0 & \text{otherwise} \end{cases}$$

and

$$Y = \begin{cases} 1 & \text{if the product of the numbers showing is even} \\ 0 & \text{otherwise} \end{cases}$$

(a) Describe the probability law of (X,Y). [8]

(b) Calculate the covariance of the components X and Y, that is, COV(X,Y). [7]

A3. Let $(\Omega, \mathcal{F}, P)$ be a probability space and A be a subset of Ω. The collection $\mathcal{G}$ of all subsets of Ω of the form $G = A \cap F$, $F \in \mathcal{F}$, is called the trace of $\mathcal{F}$ on A.

If $A \in \mathcal{F}$ and $P(A) > 0$, define a set function

Q_A on $\mathcal{G}$ by $Q_A(G) = \frac{P(G)}{P(A)}$.

Show that Q_A is a probability measure on $\mathcal{G}$. [9]

SECTION B: [60 marks]

Answer **ANY THREE** questions from this section. Each question carries 20 marks.

B4. (a) Let the joint density function of (X,Y) be given by

$$f(x,y) = \begin{cases} c(xy + y^2) \text{ if } 0 \le x \le 2, 0 \le y \le 2 \\ 0 \text{ otherwise} \end{cases}$$

where c is a constant to be determined.

Calculate $P(X + Y < 2)$. [10]

(b) The random variable X can take any of the three numbers in the set $\{1, 2, 3\}$. From the same set the random variable Y can take any number which is greater than or equal to X.

Calculate the correlation coefficient ρ_{XY} of the components X and Y. [10]

B5. (a) Two points A and B are randomly selected from the circumference of a circle, center O and radius r. Calculate the expectation of the area of triangle AOB. [10]

(b) The random variable X follows a Cauchy distribution with parameters $c \in \mathbb{R}$ and $a > 0$, so that its probability density function is given by

$$f(x) = \frac{a}{\pi[(x-c)^2 + a^2]}.$$

Calculate the characteristic function of X [10]

B6. (a) Given a sequence $\{X_k\}$; $k = 1, 2, \ldots$, of random variables, such that

- $E[X_k^2] < \alpha^2$, $\forall k = 1, 2, \ldots$
- each of the random variables in the sequence depends on one next to it.

Prove that this sequence satisfies Markov's theorem. [10]

(b) Use Chebyshev's inequality to estimate $p_k = P\{|X - \mu_X|\} \le k\sigma_X$ for $k = 1, 2, 3$ and given that $X \sim N(\mu_X, \sigma_X^2)$. [10]

B7. (a) Calculate the moment generating function of a random variable X whose probability density function is

$$f_Y(y) = \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{y-a}{b})^2}, \quad -\infty < y < \infty.$$ [7]

(b) Calculate $E(X)$ and $Var(X)$ if X is a random variable with density

$$f_X(x) = \begin{cases} \theta e^{-\theta x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$ [3,4]

(c) A random vector (X,Y) has joint density function

$$f_{X,Y}(x, y) = \begin{cases} e^{-(x+y)}, & \text{if, } x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(i) Calculate E(XY). [6]

(ii) Determine whether or not X and Y are independent. [5]

B8. (a) Calculate the characteristic function of each of the following random variables:

(i) the variable X which is normally distributed with mean μ and variance σ^2. [6]

(ii) The random variable Y with a Poisson distribution of parameter λ. [6]

(b) Given that the characteristic function of a continuous random variable X is

$$\Theta_X(t) = \frac{e^{itm}}{1 + \frac{\sigma^2 t^2}{2}},$$

find the density function of X and calculate E(X) and Var(X) as functions of m and σ. [7,3,3]

END OF QUESTION PAPER