NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF APPLIED MATHEMATICS

SMA 2106 PROBABILITY THEORY

December 2005

Time 3 hours

Candidates should attempt ALL questions in section A and any THREE questions in section B. Marks will be allocated as indicated.

SECTION A [40 marks]
Candidates should attempt **ALL** questions being careful to number them A1 to A4.

A1. (a) State and explain the three axioms of probability.
(b) Independent trials, consisting of rolling a pair of fair dice, are performed. What is the probability that an outcome of 5 appears before an outcome of 7 when the outcome of a roll is the sum of the dice? **[10 marks]**

A2. An urn contains n heliotrope and n tangerine balls. Two balls are removed from the urn together, at random.
(a) What is the sample space?
(b) What is the probability of drawing two balls of different colours?
(c) What is the probability p_n that the balls are the same colour and evaluate

$\lim_{n\to\infty} p_n$. **[10marks]**

A3. (a) Let X be a random variable with a hypergeometric distribution. Prove that

$$\frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}} \to \binom{n}{x} p^x (1-p)^{n-x}, \text{ as } N \to \infty.$$

(b) If the random variable T is the time to failure of a commercial product, show that the failure rate function is given by $Z(t) = \alpha\beta t^{\beta-1}, t > 0$ if and only if the time to failure distribution is the Weibull distribution, $f(t) = \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}}, t > 0.$

[10marks]

A4. (a) Prove that if $E_1, E_2, \ldots, E_n$ are independent events then

$$P(\bigcup_{i=1}^{n} E_i) = 1 - \prod_{i=1}^{n}[1 - P(E_i)].$$

(b) Let X be a random variable defined on a sample space S = {0, 1, 2,} with probability density function

$$P(X = x) = \begin{cases} p(1-p)^x, x \in S, 0 \le p \le 1 \\ 0, otherwise \end{cases}$$

Let $A = \{X > n\}$, $B = \{X > m\}$ and $C = \{X > m + n\}$

(i) Show that $P(X = x)$ is a discrete probability density function on S,

(ii) Show that $P(C/B) = P(A)$. **[10marks]**

SECTION B [60marks]

Candidates may attempt any **THREE** questions being careful to number them B5 through B8.

B5. (a) Prove that $M_{aX+b}(t) = e^{bt} M_X(at)$

(b) Let X be a random variable with a probability density given below

$$f(x) = \begin{cases} (\tfrac{1}{2})^x, x = 1,2,\ldots \\ 0, otherwise \end{cases}$$

(i) Find the moment generating function of X.

(ii) Use part (i) to calculate the mean and variance of X.

(iii) Deduce the characteristic function of X.

(iv) Verify the above moments using the characteristic function.

(v) Name the distribution of $f(x)$. **[20marks]**

B6. (a) State and prove Bayes' theorem.

(b) A document is equally likely to be in any of the three boxfiles. A search of the i^{th} box will discover the document (if it is indeed there) with probability p_i. What is the probability that, the document is in the first box given that I have searched

(i) the first box once and not found it?

(ii) the first box twice and not found it?

(iii) all the three boxes once and not found it.

NOTE: Assume that all the searches are independent. **[20marks]**

B7. (a) Let $E_1, E_2, \ldots, E_n$ be events.

(i) Prove that $P(E_1 \cap E_2) \ge P(E_1) + P(E_2) - 1$,

(ii) Prove that $P(\bigcup_{i=1}^{n} E_i) = 1 - P(\bigcap_{i=1}^{n} E_i')$.

(b) If the probability density function of X is given by

$$f(x) = \begin{cases} 2xe^{-x^2}, x > 0 \\ 0, otherwise \end{cases}$$

And let $Y = X^2$. Find

(i) the cumulative distribution function of Y,

(ii) the probability density function of Y,

(iii) P (Y>2),

(iv) E (Y) and Var (Y). **[20marks]**

B8. (a) Let X and Y have the joint density function

$$f(x) = \begin{cases} 2e^{-x-y}, 0 < x < y \leq \infty, \\ 0, otherwise \end{cases}$$

(i) Find the marginal distributions of X and Y.

(ii) Hence find $E(Y / X = x)$.

(b) The probability that a resident of a new residential area in Bulawayo will install a 3-metre satellite dish was estimated and found to be 0.3. Find the probability that the 10^{th} satellite dish to be installed in the area is the 5^{th} 3-metre satellite dish.

(c) State and prove Markov's inequality. **[20marks]**

END OF EXAMINATION