NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

December 2001 EXAMINATION

SMA2116 ENGINEERING MATHEMATICS II

3 Hours

Answer ALL questions in Section A and any FOUR questions in Section B.

SECTION A : Answer ALL questions from this section.

[28 Marks]

1. Evaluate $\int_{-1}^{1}\int_{0}^{|x|} x^2 y \, dy \, dx$.

[5 Marks]

2. Given the vector field $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, its magnitude r and the constant vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, find, in terms of $\mathbf{r}$, $\mathbf{a}$ and r,

   (a) grad $\left(\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\right)$,

   [3 Marks]

   (b) curl $(\mathbf{i} \times \mathbf{r})$,

   [3 Marks]

   (c) div $((\mathbf{a}\cdot\mathbf{r})\mathbf{r})$.

   [3 Marks]

3. Evaluate $\int\int\int_V x^2 dV$, where V is the interior of the sphere $x^2 + y^2 + z^2 = 9$.

[4 Marks]

4. Find the Fourier series of the function defined by

$$f(x) = x, -2\pi \le x < 2\pi.$$
$$f(x + 4\pi) = f(x).$$

[5 Marks]

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5. Evaluate $\oint_C \mathbf{F} \cdot \mathbf{dr}$, where $\mathbf{F} = 2xy\mathbf{i} - y^2\mathbf{j}$ and C is the anti-clockwise path around along the circle $x^2 + y^2 = 9$ in the xy-plane.

[5 Marks]

SECTION B - Answer any FOUR questions from this section.

[72 Marks]

6. (a) Show that, for an odd function $f(x)$ of period 2ℓ which satisfies Dirichlet's conditions,

$$E_n = \int_{-\ell}^{\ell} (f(x) - S_n(x))^2 \, dx$$

is minimised if

$$b_r = \frac{2}{\ell}\int_0^{\ell} f(x)\sin(\frac{r\pi x}{\ell})dx \qquad (\, r = 1, 2, 3...),$$

where

$$S_n(x) = \sum_{r=1}^{n} b_r \sin(\frac{r\pi x}{\ell}).$$

[6 Marks]

(b) Find the Fourier half range sine series for the function defined by

$$f(x) = x^2 \; ; \; 0 < x < 1.$$

[6 Marks]

Hence

i. Sketch the function represented by the series over the interval $-2 \leq x \leq 2$,

[3 Marks]

ii. Evaluate

$$\sum_{k=1}^{\infty}\left(\frac{1}{(2k-1)\pi} - \frac{2}{(2k-1)^3\pi^3}\right)(-1)^{k+1}.$$

[3 Marks]

7. (a) Show that, if a regular closed path, C, can be divided into two sections, C_1 and C_2, such that

on C_1 $y = f_1(x)$ $\quad x : a \to b$ and

on C_2 $y = f_2(x)$ $\quad x : b \to a$,

then, if $G(x, y)$ and $\dfrac{\partial G}{\partial y}$ are continuous,

$$\int\int_R \frac{\partial G}{\partial y}\, dy\, dx = -\int_C G\, dx,$$

where R is the region bounded by C.

[6 Marks]

(b) Verify that $\mathbf{F} = (2yz + 2x)\mathbf{i} + (2xz - 2y)\mathbf{j} + (2xy + 7)\mathbf{k}$ is a conservative and a solenoidal field.

[4 Marks]

Hence

i. find the work done in moving a particle from (0,1,0) to (2,3,3) along the path $x = 2t$, $y = 2t^2 + 1$; $z = 3t$ in this field,

[3 Marks]

ii. find a scalar potential for the vector field,

[3 Marks]

iii. find the flux of $\mathbf{F}$ through the entire surface of the sphere of radius 2 and centre the origin.

[2 Marks]

8. (a) Verify the divergence theorem for field $\mathbf{F} = x^2 z(2 - z)\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ using the cylinder $x^2 + y^2 = 9$, $\; 0 \le z \le 2$.

[9 Marks]

(b) Verify Stokes' theorem for the vector field $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + 3\mathbf{k}$ using the plane

$$y + z = 1 \; ; \; y \ge 0\,, z \ge 0, 0 \le x \le 4.$$

[9 Marks]

9. (a) Evaluate

$$\int\int_R xy\ dx\ dy,$$

where R is the region bounded by the lines $y = x$, $y = 2 - x$ and the x-axis.

[4 Marks]

(b) Evaluate

$$\int\int_R y\ dx\ dy,$$

where R is the region within the circle $x^2 + (y-2)^2 = 4$.

[5 Marks]

(c) By reversing the order of integration, evaluate

$$\int_{-\infty}^{0}\int_{0}^{e^x} \frac{1}{\ln y}\ dy\ dx$$

[5 Marks]

(d) By applying the transformation of variables $x = r\cos\theta$ and $y = 3r\sin\theta$, evaluate

$$\int\int_R xy^2\ dx\ dy,$$

where R is the region between the ellipses $9x^2 + y^2 = 9$ and $9x^2 + y^2 = 36$ in the quadrant $x < 0,\ y < 0$.

[5 Marks]

10. (a) Find the Fourier transforms of

i. $H(x-2) - H(x+2)$, where H is the unit step function,

[3 Marks]

ii. $(H(x-2) - H(x+2))\sin x$.

[4 Marks]

iii. $\delta(x+2)\sin x$, where δ is the unit impulse function.

[3 Marks]

(b) i. Use Parseval's theorem on $H(x-2) - H(x+2)$ to find an expression for π.

[3 Marks]

ii. The convolution theorem states that, if

$$g(x) = \int_{-\infty}^{\infty} f_1(t)f_2(x-t)\ dt,$$

then

$$G(\omega) = F_1(\omega)F_2(\omega).$$

Hence find the function which has as its Fourier transform the product of the transforms of $H(x-2) - H(x+2)$ and $(H(x-2) - H(x+2))\sin x$.

[5 Marks]

END OF EXAMINATION PAPER