NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

December 2002 EXAMINATION

SMA2116 ENGINEERING MATHEMATICS II

3 Hours

Answer ALL questions in Section A and any FOUR questions in Section B.

SECTION A : Answer ALL questions from this section.

[28 Marks]

1. Evaluate $\int_0^2 \int_0^{2x} xy^3 \, dy \, dx$.

[3 Marks]

2. Given the vector fields $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{F} = x\mathbf{i} + xy^2\mathbf{j} - z\mathbf{k}$, find

   (a) grad $(\mathbf{F} \cdot \mathbf{r})$,

   [3 Marks]

   (b) div $\mathbf{F} \mathbf{x} \mathbf{r}$,

   [3 Marks]

   (c) $(\mathbf{r} \cdot \nabla)\mathbf{F}$.

   [3 Marks]

3. Evaluate $\int\int\int_V x^2 dV$, where V is the interior of the cylinder $x^2+y^2 = 16$, $-2 \le z \le 4$.

[4 Marks]

4. Find the Fourier series of the function defined by

$$f(x) = \begin{cases} -2 & , \quad -\pi \le x < 0 \\ 2 & , \quad 0 \le x < \pi \end{cases}$$

$$f(x + 2\pi) = f(x).$$

[6 Marks]

5. Evaluate $\int_C \mathbf{F} \cdot \mathbf{dr}$, where $\mathbf{F} = x^2y\mathbf{i} - xy^2\mathbf{j}$ and C is the anti-clockwise path around the circle, centre the origin, radius 2.

[6 Marks]

SECTION B - Answer any FOUR questions from this section.

[72 Marks]

6. (a) Show that, for an odd function $f(x)$ of period 2ℓ which satisfies Dirichlet's conditions,

$$E_n = \int_{-\ell}^{\ell} (f(x) - S_n(x))^2 \, dx$$

is minimised if

$$b_r = \frac{2}{\ell}\int_0^{\ell} f(x)\sin(\frac{r\pi x}{\ell})dx \qquad (\ r = 0, 1, 2, 3...),$$

where

$$S_n(x) = \sum_{r=1}^{n} b_r \sin(\frac{r\pi x}{\ell}).$$

[6 Marks]

(b) Find the Fourier half range sine series for the function defined by

$$f(x) = x - 1 \ ; \ 0 < x < 2.$$

[6 Marks]

Hence

i. Sketch the function represented by the series over the interval $-4 \leq x \leq 4$,

[3 Marks]

ii. Find a series representation for π.

[3 Marks]

7. (a) Show that, if $\int_C \mathbf{F} \cdot \mathbf{dr}$ is independent of the path C, then

i. $\mathbf{F} = \text{grad}\,\phi$, where ϕ is a scalar field,

[3 Marks]

ii. $\text{curl}\,\mathbf{F} = \mathbf{0}$.

[3 Marks]

(b) Find the divergence and the curl of $\mathbf{F} = (yz+2)\mathbf{i} + (xz-3)\mathbf{j} + (xy+2z)\mathbf{k}$.

[4 Marks]

Hence, or otherwise,

i. find the work done in moving a particle from (0,0,0) to (1,3,1) along the path $y = 2x+1$; $z = x^2$ in this field,

[2 Marks]

ii. find a scalar potential for the vector field,

[3 Marks]

iii. find the flux of $\mathbf{F}$ through the entire surface of the sphere of radius 2 and centre the origin.

[2 Marks]

8. (a) Verify the divergence theorem for field $\mathbf{F} = z\mathbf{r}$ using the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$.

[9 Marks]

(b) Verify Stokes' theorem for the vector field $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ using the plane

$$y + 2z = 2 \;\; ; \;\; y \geq 0 \; , z \geq 0, 0 \leq x \leq 2.$$

[9 Marks]

9. (a) Evaluate

$$\int\int_R xy \; dx \; dy,$$

where R is the region bounded by the lines $y = x$, $y = 1 - x$ and the x-axis.

[4 Marks]

(b) Evaluate

$$\int\int_R x^2 \; dx \; dy,$$

where R is the region within the semicircle $x^2 + y^2 = 4$, $y < 0$.

[4 Marks]

(c) By reversing the order of integration, evaluate

$$\int_{-\infty}^{0} \int_{0}^{e^x} \frac{1}{\ln y} \, dy \, dx \quad .$$

[5 Marks]

(d) By applying the transformation of variables $x = r\cos\theta$ and $y = 3r\sin\theta$, evaluate

$$\int\int_R xy \, dx \, dy,$$

where R is the region between the ellipses $9x^2 + y^2 = 9$ and $9x^2 + y^2 = 36$ in the quadrant $x > 0, \ y < 0$.

[5 Marks]

10. (a) Find the Fourier transforms of

i. $H(x+2) - H(x-2)$, where H is the unit step function,

[4 Marks]

ii. $\delta(x-1)\cos x$, where δ is the unit impulse function.

[2 Marks]

iii. $H(x-1) - H(x)$.

[4 Marks]

(b) i. Use Parseval's theorem on $H(x-1) - H(x)$ to find an expression for π.

[3 Marks]

ii. The convolution theorem states that, if

$$g(x) = \int_{-\infty}^{\infty} f_1(t) f_2(x-t) \, dt,$$

then

$$G(\omega) = F_1(\omega)F_2(\omega).$$

Hence find the function which has as its Fourier transform the product of the transforms of $H(x+2) - H(x-2)$ and $H(x-1) - H(x)$.

[5 Marks]

END OF EXAMINATION PAPER