NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

July 2003 Supplementary Examination

SMA2116 ENGINEERING MATHEMATICS II

3 Hours

Answer ALL questions in Section A and any FOUR questions in Section B.

SECTION A : Answer ALL questions from this section.

[28 Marks]

1. Evaluate $\int_0^1 \int_0^x x^2y^3 \, dy \, dx$.

[4 Marks]

2. Given the vector fields $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{F} = x^2\mathbf{i} + 2xy\mathbf{j} - 2xz\mathbf{k}$, find

   (a) $\text{grad}\,(\mathbf{F} \cdot \mathbf{r})$

   [2 Marks]

   (b) $\text{curl}\,\mathbf{F}$

   [2 Marks]

   (c) $(\mathbf{r} \cdot \nabla)\mathbf{F}$.

   [3 Marks]

3. Evaluate $\int\int\int_V x^2 dV$, where V is the interior of the cylinder $x^2 + y^2 = 9,\ 0 \leq z \leq 2$.

[5 Marks]

4. Find the Fourier series of the function defined by

$$f(x) = \begin{cases} 0 & , \quad -\pi \leq x < 0 \\ 1 & , \quad 0 \leq x < \pi \end{cases}$$

$$f(x + 2\pi) = f(x).$$

[6 Marks]

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5. Evaluate $\int_C \mathbf{F} \cdot \mathbf{dr}$, where $\mathbf{F} = xy\mathbf{i} - y^2\mathbf{j}$ and C is the path along the straight line from (0,0) to (1,2), and then along the curve $y = x^2 + 1$ from (1,2) to (2,5).

[6 Marks]

SECTION B - Answer any FOUR questions from this section.

[72 Marks]

6. (a) Show that, for an even function $f(x)$ of period 2ℓ which satisfies Dirichlet's conditions,

$$E_n = \int_{-\ell}^{\ell} (f(x) - S_n(x))^2 \, dx$$

is minimised if

$$a_r = \frac{2}{\ell}\int_0^{\ell} f(x)\cos(\frac{r\pi x}{\ell})dx \qquad (r = 0, 1, 2, 3...),$$

where

$$S_n(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} a_r \cos(\frac{r\pi x}{\ell}).$$

[6 Marks]

(b) Find the Fourier half range cosine series for the function defined by

$$f(x) = x \; ; \; 0 < x < 2.$$

[6 Marks]

Hence

i. Sketch the function represented by each of the series over the interval $-4 \leq x \leq 4$, [4 Marks]

ii. Find a series representation for π^2. [2 Marks]

7. (a) Show that, if $\int_C \mathbf{F} \cdot \mathbf{dr}$ is independent of the path C, then

i. $\mathbf{F} = \text{grad}\,\phi$, where ϕ is a scalar field, [3 Marks]

ii. $\text{curl}\,\mathbf{F} = 0$. [3 Marks]

(b) Verify that $\mathbf{F} = (yz + 2z + x + 1)\mathbf{i} + (xz - y)\mathbf{j} + (xy + 2x + 1)\mathbf{k}$ is a conservative and a solenoidal field.

[4 Marks]

Hence

i. find the work done in moving a particle from (0,0,0) to (1,4,1) along the path $y = 3x + 1$; $z = x^3$ in this field,

[2 Marks]

ii. find a scalar potential for the vector field,

[3 Marks]

iii. find the flux of $\mathbf{F}$ through the entire surface of the cube of side 1, centre the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

[2 Marks]

8. (a) Verify the divergence theorem for field $\mathbf{F} = z^2\mathbf{r}$ using the sphere $x^2 + y^2 + z^2 = 4$.

[9 Marks]

(b) Verify Stokes' theorem for the vector field $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ using the plane

$$y + z = 1 \;\; ; \;\; y \geq 0 \, , z \geq 0, 0 \leq x \leq 2.$$

[9 Marks]

9. (a) Evaluate

$$\int\int_R xy^2 \, dx \, dy,$$

where R is the region bounded by the lines $y = x$, $y = 1 - x$ and the y-axis.

[4 Marks]

(b) Evaluate

$$\int\int_R y^2 \, dx \, dy,$$

where R is the region within the semicircle $x^2 + y^2 = 4$, $y > 0$.

[5 Marks]

(c) By reversing the order of integration, evaluate

$$\int_{-\infty}^{0} \int_{0}^{e^x} \frac{1}{lny}\, dy\, dx \quad .$$

[4 Marks]

(d) By applying the transformation of variables $x = r\cos\theta$ and $y = 3r\cos\theta$, evaluate

$$\int\int_R xy\, dx\, dy,$$

where R is the region between the ellipses $9x^2 + y^2 = 9$ and $9x^2 + y^2 = 81$ in the quadrant $x < 0,\ y > 0$.

[5 Marks]

10. (a) Find the Fourier transforms of

i. $H(x + \pi) - H(x - \pi)$, where H is the unit step function,

[3 Marks]

ii. $(H(x + \pi) - H(x - \pi))\cos x$.

[3 Marks]

iii. $\delta(x - \pi)\cos x$, where δ is the unit impulse function.

[3 Marks]

(b) i. Use Parseval's theorem on $H(x + \pi) - H(x - \pi)$ to find an expression for π.

[4 Marks]

ii. The convolution theorem states that, if

$$g(x) = \int_{-\infty}^{\infty} f_1(t) f_2(x - t)\, dt,$$

then

$$G(\omega) = F_1(\omega) F_2(\omega).$$

Hence find the function which has as its Fourier transform the product of the transforms of $H(x + \pi) - H(x - \pi)$ and $(H(x + \pi) - H(x - \pi))\cos x$.

[5 Marks]

END OF EXAMINATION PAPER