NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

August 2004 SUPPLEMENTARY EXAMINATION

SMA2116 ENGINEERING MATHEMATICS II

3 Hours

Answer ALL questions in Section A and any FOUR questions in Section B.

SECTION A : Answer ALL questions from this section.

[28 Marks]

1. Evaluate $\int\int_R 2x\, dy\, dx$, where R is the region contained by the x axis, the y axis and the line $y = 1 - x$.

[5 Marks]

2. Given the vector field $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, its magnitude r and the constant vector $\mathbf{a} = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, find, in terms of $\mathbf{r}$, $\mathbf{a}$ and r,

   (a) grad $(\mathbf{a} \cdot \mathbf{r})$,

   [2 Marks]

   (b) curl $\mathbf{axr}$,

   [4 Marks]

   (c) div $(\mathbf{axr})$.

   [2 Marks]

3. Evaluate $\int\int\int_V x^2 dV$, where V is the interior of the cylinder $x^2 + y^2 = 4$, $0 < z < 3$.

[4 Marks]

4. Find the Fourier half range sine series for the function defined by

$$f(x) = 3, 0 \leq x < \pi.$$

[6 Marks]

---

5. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = xy\mathbf{i} + x\mathbf{j}$ and C is the path along $y = 1 + x^2$ from $(0,1)$ to $(2,5)$.

[5 Marks]

SECTION B - Answer any FOUR questions from this section.

[72 Marks]

6. (a) Show that, for an odd function $f(x)$ of period 2ℓ which satisfies Dirichlet's conditions,

$$E_n = \int_{-\ell}^{\ell} \left(f(x) - S_n(x)\right)^2 \, dx$$

is minimised if

$$b_r = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{r\pi x}{\ell}\right)dx \qquad (r = 0, 1, 2, 3...),$$

where

$$S_n(x) = \sum_{r=1}^{n} a_r \sin\left(\frac{r\pi x}{\ell}\right).$$

[6 Marks]

(b) Find the Fourier series for the function defined by

$$f(x) = x\,; \quad -2 < x \leq 2$$
$$f(x+4) = f(x).$$

[6 Marks]

Hence

i. Sketch the function represented by the series over the interval $-6 \leq x \leq 6$,

[3 Marks]

ii. Find a series representation for π.

[3 Marks]

7. (a) Show that, if $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path C, then

i. $\mathbf{F} = \operatorname{grad}\phi$, where ϕ is a scalar field,

[3 Marks]

ii. $\operatorname{curl}\mathbf{F} = \mathbf{0}$.

[3 Marks]

(b) For the vector field $\mathbf{F} = (yz + 2x)\mathbf{i} + (xz - y)\mathbf{j} + (xy - z)\mathbf{k}$, find $\operatorname{curl}\mathbf{F}$ and $\operatorname{div}\mathbf{F}$.

[4 Marks]

Hence

i. find the work done in moving a particle from (0,0,1) to (1,2,3) along the path $x = t^2$, $y = 2t$; $z = 2t + 1$ in this field,

[2 Marks]

ii. find, if possible, a scalar potential for the vector field,

[4 Marks]

iii. find the flux of $\mathbf{F}$ through the entire surface of the sphere of radius 1 and centre the origin.

[2 Marks]

8. (a) Verify the divergence theorem for field $\mathbf{F} = (y\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ using the sphere $x^2 + y^2 + z^2 = 4$.

[9 Marks]

(b) Verify Stokes' theorem for the vector field $\mathbf{F} = y\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ using the plane

$$y + z = 1 \;\; ; \;\; y \geq 0 \, , z \geq 0, 0 \leq x \leq 2.$$

[9 Marks]

9. (a) Evaluate

$$\int\int_R xy \; dx \; dy,$$

where R is the region bounded by the lines $y = 2 - x$, $y = 1 - x$, $y = x$ and the x-axis.

[4 Marks]

(b) Evaluate

$$\int\int_R y^2 \; dx \; dy,$$

where R is the region within the semi-circle $x^2 + y^2 = 1, y > 0$.

[4 Marks]

(c) By reversing the order of integration, evaluate

$$\int_{-\infty}^{1} \int_{0}^{e^x} \frac{1}{1 - \ln y} \, dy \, dx \quad .$$

[5 Marks]

(d) By applying the transformation of variables $x = r\cos\theta$ and $y = 2r\sin\theta$, evaluate

$$\int \int_R xy \, dx \, dy,$$

where R is the region between the ellipses $4x^2 + y^2 = 1$ and $4x^2 + y^2 = 4$ in the quadrant $x < 0, \; y < 0$.

[5 Marks]

10. (a) Find the Fourier transforms of

i. $H(x+2) - H(x-2)$, where H is the unit step function,

[4 Marks]

ii. $H(x-2)e^{-2x}$.

[4 Marks]

(b) i. Use Parseval's theorem on $H(x+2) - H(x-2)$ to find an expression for π.

[3 Marks]

ii. The convolution theorem states that, if

$$g(x) = \int_{-\infty}^{\infty} f_1(t) f_2(x-t) \, dt,$$

then

$$G(\omega) = F_1(\omega) F_2(\omega).$$

Hence find the function which has as its Fourier transform the product of the transforms of $H(x+2) - H(x-2)$ and $H(x-2)e^{-2x}$.

[7 Marks]

END OF EXAMINATION PAPER