NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

December 2004 EXAMINATION

SMA2116 ENGINEERING MATHEMATICS II

3 Hours

Answer ALL questions in Section A and any FOUR questions in Section B.

SECTION A : Answer ALL questions from this section.

[28 Marks]

1. Evaluate $\int_0^1 \int_0^{(1-x)} xy \; dydx.$

[4 Marks]

2. Given the vector field $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, its magnitude r and the constant vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, find, in terms of r and a,

   (a) grad $(\mathbf{a} \cdot \mathbf{r})$,

   [3 Marks]

   (b) curl $(\mathbf{axr})$,

   [4 Marks]

   (c) div $(\mathbf{a} \cdot \mathbf{r})\mathbf{r}$.

   [3 Marks]

3. Evaluate $\int\int\int_V x^2 dV$, where V is the interior of the cylinder $x^2 + y^2 = 9, \quad 0 \leq z \leq 4$.

[4 Marks]

4. Find the Fourier series of the function defined by

$$f(x) = -x, -\pi \leq x < \pi.$$
$$f(x + 2\pi) = f(x).$$

[5 Marks]

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5. Evaluate $\oint_C \mathbf{F} \cdot \mathbf{dr}$, where $\mathbf{F} = xy\mathbf{i} - x\mathbf{j}$ and C is the anti-clockwise path around along the circle $x^2 + y^2 = 4$ in the xy-plane.

[5 Marks]

SECTION B - Answer any FOUR questions from this section.

[72 Marks]

6. (a) Show that, for an odd function $f(x)$ of period 2ℓ which satisfies Dirichlet's conditions,

$$E_n = \int_{-\ell}^{\ell} (f(x) - S_n(x))^2 \, dx$$

is minimised if

$$b_r = \frac{2}{\ell} \int_0^{\ell} f(x) \sin(\frac{r\pi x}{\ell}) dx \qquad (r = 1, 2, 3...),$$

where

$$S_n(x) = \sum_{r=1}^{n} b_r \sin(\frac{r\pi x}{\ell}).$$

[6 Marks]

(b) Find the Fourier half range sine series for the function defined by

$$f(x) = 2 \; ; \; 0 < x < 1.$$

[6 Marks]

Hence

i. Sketch the function represented by the series over the interval $-2 \leq x \leq 2$,

[3 Marks]

ii. Find a series representation for π.

[3 Marks]

7. (a) Show that, if $\mathbf{F} = grad\phi$, then $\mathbf{F}$ is irrotational and conservative.

[6 Marks]

(b) Verify that $\mathbf{F} = (2yz + 2x)\mathbf{i} + (2xz - 2y)\mathbf{j} + (2xy + 7)\mathbf{k}$ is an irrotational and a solenoidal vector field.

[4 Marks]

Hence

i. find the work done in moving a particle from (0,1,0) to (2,3,3) along the path $x = 2t$, $y = 2t + 1$; $z = 3t$ in this field,

[3 Marks]

ii. find a scalar potential for the vector field,

[3 Marks]

iii. find the flux of **F** through the entire surface of the sphere of radius 4 and centre the origin.

[2 Marks]

8. (a) Verify the divergence theorem for field $\mathbf{F} = (4x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ using the sphere $x^2 + y^2 + z^2 = 9$.

[9 Marks]

(b) Verify Stokes' theorem for the vector field $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + \mathbf{k}$ using the plane

$$y + z = 1 \ ; \ y \geq 0 \ , z \geq 0, 0 \leq x \leq 4.$$

[9 Marks]

9. (a) Evaluate

$$\int\int_R xy \, dx \, dy,$$

where R is the region bounded by the lines $y = x$, $y = 2 - x$ and the y-axis.

[4 Marks]

(b) By changing to polar coordinates, evaluate

$$\int_0^1 \int_x^{\sqrt{2-x^2}} x \, dy \, dx.$$

[5 Marks]

(c) By reversing the order of integration, evaluate

$$\int_{-1}^{1}\int_{|x|}^{1} y \, dy \, dx.$$

[4 Marks]

(d) By applying the transformation of variables $x = r\cos\theta$ and $y = 3r\sin\theta$, evaluate

$$\int\int_R xy \, dx \, dy,$$

where R is the region enclosed by the ellipse $9x^2 + y^2 = 36$ in the quadrant $x < 0,\ y > 0$.

[5 Marks]

10. (a) Find the Fourier transforms of

i. $H(x-2) - H(x+2)$, where H is the unit step function,

[4 Marks]

ii. $H(x-2)e^{-x}$.

[4 Marks]

(b) i. Use Parseval's theorem on $H(x-2) - H(x+2)$ to find an expression for π.

[4 Marks]

ii. The convolution theorem states that, if

$$g(x) = \int_{-\infty}^{\infty} f_1(t) f_2(x-t) \, dt,$$

then

$$G(\omega) = F_1(\omega)F_2(\omega).$$

Hence find the function which has as its Fourier transform the product of the transforms of $H(x-2) - H(x+2)$ and $H(x-2)e^{-x}$.

[6 Marks]

END OF EXAMINATION PAPER