NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

2005 SUPPLEMENTARY EXAMINATION

SMA2116 ENGINEERING MATHEMATICS II

3 Hours

Answer ALL questions in Section A and any FOUR questions in Section B.

SECTION A : Answer ALL questions from this section.

[28 Marks]

1. Evaluate $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2 + y^2 \; dx\, dy.$

[4 Marks]

2. For the vector field $\mathbf{P} = xy\mathbf{i} - x^2\mathbf{j}, x \neq 0,$

   (a) determine $\operatorname{div}\mathbf{P}$,

   [2 Marks]

   (b) determine $\operatorname{curl}\mathbf{P}$,

   [2 Marks]

   (c) sketch the vector field.

   [3 Marks]

3. Deduce the flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$ over the surface of the plane $x + y + z = 1,\; x \geq 0, y \geq 0, z \geq 0.$

[6 Marks]

4. Find the Fourier series of the function defined by

$$f(x) = x, -\pi < x \leq \pi$$
$$f(x + 2\pi) = f(x).$$

[5 Marks]

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5. Evaluate $\oint_C \mathbf{F} \cdot \mathbf{dr}$, where $\mathbf{F} = (x+y)\mathbf{i} + (2y-x)\mathbf{j}$ and C is the path along the straight line from $(0,0)$ to $(1,0)$, then along the curve $y = x^2 - x$ from $(1,0)$ to $(0,0)$.

[6 Marks]

SECTION B - Answer any FOUR questions from this section.

[72 Marks]

6. (a) Show that, for an odd function $f(x)$ of period 2ℓ which satisfies Dirichlet's conditions,

$$E_n = \int_{-\ell}^{\ell} (f(x) - S_n(x))^2 \, dx$$

is minimised if

$$b_r = \frac{2}{\ell}\int_0^{\ell} f(x)\sin(\frac{r\pi x}{\ell})dx \qquad (\, r = 1, 2, 3...),$$

where

$$S_n(x) = \sum_{r=1}^{\infty} b_r \sin(\frac{r\pi x}{\ell}).$$

[5 Marks]

(b) Find the Fourier half range cosine series and the Fourier half range sine series for the function defined by

$$f(x) = x \; ; \; 0 < x < 1.$$

[8 Marks]

Hence

i. Sketch the function represented by each of the series over the interval $-2 \leq x \leq 2$,

[4 Marks]

ii. Find a series representation for π^2.

[2 Marks]

7. (a) i. Show that, if $\mathbf{F} = \text{grad}\,\phi$, where ϕ is continuous with continuous first and second partial derivatives, then $\text{curl}\,\mathbf{F} = \mathbf{0}$.

[3 Marks]

ii. Show that, if $\int_C \mathbf{F} \cdot \mathbf{dr}$ is independent of the path C from point P_0 to the point P_1, then $\mathbf{F} = \text{grad}\,\phi$.

[3 Marks]

(b) Verify that $\mathbf{F} = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$ is a conservative vector field.

[5 Marks]

Hence

i. find a scalar potential for the vector field,

[4 Marks]

ii. find the work done in moving a particle from (1,0,0) to (0,1,1) along the path

$$x = \cos t, y = \sin t, z = \frac{t}{\pi}; t : 0\pi$$

in this field.

[3 Marks]

8. (a) Verify the divergence theorem for field $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ using the cylinder $x^2 + y^2 = 1, 0 \leq z \leq 1$.

[9 Marks]

(b) Verify Stokes' theorem for the vector field $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ using the surface of the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$.

[9 Marks]

9. (a) By reversing the order of integration, evaluate

$$\int_0^2 \int_x^{6-2x} \frac{1}{y}\, dy\, dx.$$

[4 Marks]

(b) By means of the substitutions $x = 2r\cos\theta$ and $y = r\sin\theta$, evaluate

$$\int\int_R \sqrt{4 - x^2 - 4y^2}dx\ dy,$$

where R is the area inside the ellipse $x^2+4y^2 = 4$ and in the quadrant $(x \leq 0,\ y \geq 0)$.

[5 Marks]

(c) Evaluate $\int\int_R x^2\, dx\, dy$, where R is the area inside the circle $x^2 + y^2 = 2$

[5 Marks]

(d) Evaluate $\int\int\int_V z^2\, dx\, dy\, dz$, where V is the interior of the sphere $x^2 + y^2 + z^2 = 16$.

[4 Marks]

10. (a) Find the Fourier transforms of

i. $H(x + k) - H(x - k)$, where H is the unit step function and k is a positive constant,

ii. $H(x - k)e^{-kx}$.

[3 Marks]

iii. $\delta(x - k)\cos kx$, where δ is the unit impulse function.

[3 Marks]

(b) i. Use Parseval's theorem on $H(x + k) - H(x - k)$ to show that

$$\pi = 2\int_0^\infty \frac{\sin^2\omega}{\omega^2}\, d\omega.$$

[3 Marks]

ii. The convolution theorem states that, if

$$g(x) = \int_{-\infty}^{\infty} f_1(t)f_2(x - t)\, dt,$$

then

$$G(\omega) = F_1(\omega)F_2(\omega).$$

Hence find and sketch the function which has as its Fourier transform the product of the transforms of $H(x + k) - H(x - k)$ and $H(x - k)e^{-kx}$.

[4 Marks]

[5 Marks]

END OF EXAMINATION PAPER