NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 2116

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA2116: ENGINEERING MATHEMATICS II

DECEMBER 2005 EXAMINATIONS

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY FOUR** questions from Sections B.

SECTION A: Answer ALL questions in this section [28].

A1. Show that $\nabla^2 (\ln r) = \frac{1}{r^2}$, where r has its usual meaning. [4]

A2. Convert the integral

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \int_{0}^{x} (x^2 + y^2) dz dx dy$$

to an equivalent integral in cylindrical coordinates and evaluate the result. [5]

A3. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = y\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and

(a) C is the path $\mathbf{r} = t\mathbf{i} + 2t\mathbf{j} - (t+1)\mathbf{k}$ from $(1, 2, -2)$ to $(3, 6, -4)$; [3]

(b) C is the path in the xy-plane in the anti-clockwise direction along the ellipse $x^2 + 4y^2 = 4$ from $(-2, 0)$ to $(0, 1)$. [6]

A4. Find the Fourier series for the function defined by

$$f(x) = \pi - x, \qquad -\pi < x \leq \pi$$

$$f(x + 2\pi) = f(x).$$

[5]

A5. Given that $\mathbf{F} = x\mathbf{r}$, find the flux of F through the plane surface

$$x + 2y + z = 2, \quad x > 0, y > 0, z > 0.$$

[5]

SECTION B: Answer any FOUR questions in this section [72].

B6. (a) By reversing the order of integration, evaluate

$$\int_0^{\infty} \int_{\tan^{-1} y}^{\frac{\pi}{2}} \cot x dx dy.$$

[4]

(b) By means of the substitutions $x = r\cos\phi$ and $y = 2r\sin\phi$, evaluate

$$\int\int_R y dx dy,$$

where R is the region in the positive quadrant, i.e. $x > 0$ and $y > 0$, between the ellipses $4x^2 + y^2 = 4$ and $4x^2 + y^2 = 64$. [5]

(c) By changing to polar coordinates, evaluate

$$\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} \sqrt{4 - x^2 - y^2} dy dx.$$

[5]

(d) Evaluate $\int\int\int_V y^2 dV$, where V is the interior of the cylinder $x^2 + y^2 = 4, \ 0 \leq z \leq 3.$ [4]

B7. (a) Verify the divergence (Gauss) theorem for the field $\mathbf{F} = x\mathbf{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, using the cylinder $x^2 + y^2 = 4, 0 < z < 1.$ [9]

(b) Verify Stokes' theorem for the field $\mathbf{F} = x\mathbf{r}$ over the curved surface of the hemisphere $x^2 + y^2 + z^2 = 4, \quad x > 0, y > 0, z > 0$. [9]

B8. (a) (i) Show that, if $\mathbf{F} = \nabla\phi$, where ϕ is continuous with first and second partial derivatives, then $\nabla \times \mathbf{F} = 0$. [3]

(ii) If $\mathbf{F} = \nabla\phi$, where ϕ is single-valued and has continuous partial derivatives, show that the work done in moving a particle from one point $P_1\,(x_1, y_1, z_1)$ in this force field to another point $P_2\,(x_2, y_2, z_2)$ is independent of the path joining the two points. [4]

(b) Verify that $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^3}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, is a conservative vector field. [5]

Hence

(i) Find a scalar potential for the vector field, [4]

(ii) Find the work done in moving a particle from $(1, 0, 0)$ to $(0, 1, 1)$ along the path

$$x = \cos t, y = \sin t, z = \frac{t}{\pi}; 0 \leq t \leq \pi$$

in this force field. [2]

B9. (a) Show that, for an even function $f(x)$ of period $2l$ which satisfies Dirichlet's conditions,

$$E_n = \int_{-l}^{l} (f(x) - S_n(x))^2\, dx$$

is minimised if

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

and

$$a_r = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{r\pi x}{l}\right) dx, \quad (r = 1, 2, 3, ...)$$

where

$$S_n(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} a_r \cos\left(\frac{r\pi x}{l}\right).$$

[3+5]

(b) Find the Fourier series of the sawtooth wave

$$f(x) = |x| \quad for - 1 < x < 1$$

$$f(x + 2) = f(x).$$

[6]

Hence

(i) Sketch the function represented by the series over the interval

$$-3 \leq x \leq 3,$$

[2]

(ii) Find a series representation for π^2. [2]

B10. (a) Find the Fourier transforms of

(i) $H(x+1) - H(x-1)$, where H is the unit step function, [3]

(ii) $H(x)e^{-x}$. [3]

(iii) $\delta(x-1)e^{-x}$, where δ is the unit impulse function. [3]

(b) (i) Use Parseval's theorem on $H(x+1) - H(x-1)$ to show that

$$\pi = 2\int_0^\infty \frac{\sin^2 w}{w^2} dw.$$

[4]

(ii) The convolution theorem states that, if

$$g(x) = \int_{-\infty}^{\infty} f_1(t) f_2(x-t) dt,$$

then

$$G(w) = F_1(w) F_2(w).$$

Hence find and sketch the function which has as its Fourier transform the product of the transforms of $H(x+1) - H(x-1)$ and $H(x)e^{-x}$. [5]

END OF QUESTION PAPER