NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 2201

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

COMPLEX ANALYSIS

MAY 2002

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY FOUR** questions from Section B.

SECTION A

A1. (a) State Liouville's theorem.

(b) Let $f(z)$ and $g(z)$ be entire functions such that $|f(z)| > |g(z)| \forall z \in \mathbf{C}$. Prove that $f(z) = cg(z)$ for some constant c.

[2,3]

A2. Find the Laurent series of the function

$$\frac{z}{z^2 - 2z + 1}$$

about the pole $z = 1$. What is the multiplicity of the pole? What is the residue at the pole? [5]

A3. State Cauchy's Integral formula for the nth derivative of an analytic function, and hence evaluate the integral

$$\oint \frac{ze^{4z}}{[illegible]} dz$$

A4. Given that

$$U(x,y) = e^{3-2xy} \cdot \cos(x^2 - y^2)$$

is the real part of an analytic function $f(z) = u + iv$. Find the function $v(x,y)$. Express f as a function of $z = x + iy$ [8]

A5. Find the image of the interior of the circle $C : |z-2| = 2$ under the bilinear transformation $w = f(z) = \frac{z}{2z-8}$ [7]

A6. Prove that the equation

$$z + 3 + 2e^z = 0$$

has precisely one root in the left - plane. [8]

SECTION B

B7. (a) Establish the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

for an analytic function $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$ [5]

(b) Let A be an open subset of $\mathbf{C}$ and

$$A^* = \{z : z^* \in A\}$$

. Suppose f is analytic on A and define a function g on A^* by $g(z) = f(\bar{z})$. Show that g is analytic on A^*. [6]

(c) If $u(z)$ is harmonic and $v(z)$ is harmonic, then

(i) Is $u(z) + v(z)$ harmonic?

(ii) Is $u(z) \cdot v(z)$ harmonic?

(iii) Is $u(v(z))$ harmonic?

[4]

B8. Locate the singularities of the function

$$f(z) = \frac{(z^2+1)^2}{25z^3 - 4z(z^2+1)^2}$$

and find the residue at each singularity. Hence show that

B9. If $f(z)$ has a pole of order n at $z = a$, prove that

$$\text{Res} f(z)|_{z=a} = \frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]$$

If $f(z) = \frac{z}{(z-1)(z+2)^2}$ find the residue of $f(z)$ at $z = -2$. Determine the Laurent expansion for $f(z)$ about the point $z = -2$, starting where each expansion is valid. Evaluate $\oint_c f(z)dz$ where C is the circle $|z+3| = 2$ described anticlockwise. [15]

B10. Show that the general linear fractional transformation

$$w = \frac{az+b}{cz+d}$$

may be written in terms of just three arbitrary constatnts and deduce that the general formula in terms of three points (z_1, z_2, z_3) and their images (w_1, w_2, w_3) is

$$\frac{(w-w_1)(w_3-w_1)}{(w-w_2)(w_3-w_1)} = \frac{(z-z_1)(z_3-z_1)}{(z-z_2)(z_3-z_1)}.$$

Find the linear fractional transformation which maps $(z_1, z_2, z_3) = (-1, 1, 2)$ onto $(w_1, w_2, w_3) = (0, -1, -3)$.

Find also, for this mapping, the image of the right-hand half $(x > 0)$ of the interior of the unit circle $|z| = 1$ Draw a rough sketch to illustrate your answer. [15]

B11. (a) State the Maximum Modulus theorem.

(b) Let f be analytic and let $f'(z) \neq 0$. Given $\epsilon > 0$, show that there exists a $z \in A$ and an $s \in A$ such that $|z - z_0| < \epsilon$, $|s - s_0| < \epsilon$ and

$$|f(z)| > |f(z_0)|, |f(s)| < |f(s_0)|$$

(c) Let f be analytic and nonzero in a region A. Show that $|f|$ has no local minima in A. If f has zeros in A, show that by example that this conlusion does not hold.

[3,6,6]

B12. (a) State Rouche's Theorem.

(b) Show that the equation

$$z + e^{-z} = \lambda, \lambda > 1$$

has only one root in the right half-plane $\text{Re}(z) > 0$.

(c) Show that if f is analytic inside and on the unit cirle $|z| = 1$, and $0 < |f(z)| < 1$ when $|z| = 1$, then f has exactly one fixed point inside the unit circle.

(d) Show that the equation

$$4z^3 - 12z^2 + 2z + 10 = 0$$

has two roots inside the annulus $1/2 < |z - 1| < 2$