NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 2201

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

COMPLEX ANALYSIS

MAY 2003

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY FOUR** questions from Section B.

SECTION A

A1. (a) State Liouville's theorem.

(b) Let $f(z)$ be an entire function such that

$$|f(z) + \sin z| > 10, \forall z \in \mathbb{C}$$

Find an expression for f and hence show that f is unbounded.

[2,3]

A2. Find the Laurent series of the function

$$\frac{z}{z^2 + 3z + 2}$$

about the pole $z = -1$. What is the multiplicity of the pole? Give region of convergence of the series. [5]

A3. State Cauchy's Integral formula for the nth derivative of an analytic function, and hence evaluate the integral

$$\oint_C \frac{5z^2 - 3z + 2}{(z-1)^3} dz$$

where C is any simple closed curve enclosing $z = 1$. What is the value of the integral if C is the circle $|z| = 1/2$? [7]

A4. Given that

$$U(x,y) = e^{3-2xy} \cdot \cos(x^2 + y^2)$$

is the real part of an analytic function $f(z) = u + iv$. Find the function $v(x,y)$. Express f as a function of $z = x + iy$ [8]

A5. Find the image of the interior of the circle $C : |z+2| = 2$ under the bilinear transformation $w = f(z) = \frac{z}{2z-8}$ [7]

A6. Find $f(z)$ such that $f'(z)$ is equal to $4z - 3$ and $f(1+i) = -3i$ [8]

SECTION B

B7. (a) State the Maximum Modulus theorem.

(b) Let f be analytic and let $f'(z) \neq 0$. Given $\epsilon > 0$, show that there exists a $z \in A$ and an $s \in A$ such that $|z - z_0| < \epsilon$, $|s - s_0| < \epsilon$ and

$$|f(z) > |f(z_0)|, |f(s)| < |f(s_0)|$$

(c) Let f be analytic and nonzero in a region A. Show that $|f|$ has no local minima in A. If f has zeros in A, show that by example that this conlusion does not hold.

[3,6,6]

B8. (a) State Rouche's Theorem.

(b) Show that the equation

$$z + e^{-z} = \lambda, \lambda > 1$$

has only one root in the right half-plane $\text{Re}(z) > 0$.

(c) Show that if f is analytic inside and on the unit cirle $|z| = 1$, and $0 < |f(z)| < 1$ when $|z| = 1$, then f has exactly one fixed point inside the unit circle.

(d) Show that the equation

$$4z^3 - 12z^2 + 2z + 10 = 0$$

has two roots inside the annulus $1/2 < |z - 1| < 2$

[2,4,5,4]

"LIBRARY USE ONLY"

B9. (a) Establish the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

for an analytic function $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$ [3]

(b) Prove that $f(z) = \frac{1}{z-2}$ is analytic in any region not including $z = 2$. [5]

(c) If $v(x,y) = x^2 + 4x - y^2 + 2y$,

(i) Show that $v(x,y)$ is harmonic, [2]

(ii) Hence find $u(x,y)$ [3]

(iii) Express $f(x,y) = u(x,y) + iv(x,y)$ as $f(z)$. [2]

B10. Locate the singularities of the function

$$f(z) = \frac{(z^2+1)^2}{25z^3 - 4z(z^2+1)^2}$$

and find the residue at each singularity. Hence show that

$$\int_0^{2\pi} \frac{\cos^2\theta}{17 - 8\cos 2\theta} d\theta = \frac{\pi}{12}$$

[15]

B11. If $f(z)$ has a pole of order n at $z = a$, prove that

$$\text{Res} f(z)|_{z=a} = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)]$$

If $f(z) = \frac{z^2}{(z-2)(z^2+1)}$ find the residue of $f(z)$ at $z = 2, i, -i$. Determine the Laurent expansion for $f(z)$ about the point $z = -2$, starting where each expansion is valid. Evaluate $\oint_c f(z)dz$ where C is the circle $|z+3| = 2$ described anticlockwise. [15]

B12. Let $w = z^2$ define a transformation from the z - plane (xy plane) (uv plane). Consider a triangle in the z plane with vertices at A(2,1), B(4,1), C(4,3).

(a) Show that the *image or mapping* of this triangle is a curvilinear triangle in the uv plane.

(b) Find the angles of this curvilinear triangle and compare with those of the original triangle.

[15]

END OF QUESTION PAPER