NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 2201

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

COMPLEX ANALYSIS SUPPLEMENTARY

JULY 2003

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY FOUR** questions from Section B.

SECTION A: Answer ALL questions in this section [40].

A1. (a) State Liouville's theorem.

(b) Let $f(z)$ be an entire function such that $|f(z) + \sin z| > 10 \forall z \in \mathbb{C}$. Find an expression for f and hence show that f is unbounded. [2,3]

A2. Find the Laurent series of the function

$$\frac{z}{(z+1)(z+2)}$$

about the pole $z = -1$. What is the multiplicity of the pole? Give region of convergence of the series. [5]

A3. State Cauchy's Integral formula for the nth derivative of an analytic function, and hence evaluate the integral

$$\oint_C \frac{e^z}{z(z+1)} dz$$

where C is the circle $|z-1| = 3$. What is the value of the integral if C is the circle $|z| = 1/2$? [7]

A4. Given that

$$U(x,y) = e^{3-2xy} \cdot \cos(x^2 - y^2)$$

is the real part of an analytic function $f(z) = u+iv$. Find the function $v(x,y)$. Express f as a function of $z = x + iy$ [8]

A5. Find the image of the interior of the circle $C : |z+2| = 2$ under the bilinear transformation $w = f(z) = \frac{z}{2z-8}$ [7]

A6. Find $f(z)$ such that $f'(z)$ is equal to $4z - 3$ and $f(1+i) = -3i$ [8]

SECTION B: Answer THREE questions in this section [60].

B7. (a) State the Maximum Modulus theorem.

(b) Let f be analytic and let $f'(z) \neq 0$. Given $\epsilon > 0$, show that there exists a $z \in A$ and an $s \in A$ such that $|z - z_0| < \epsilon$, $|s - s_0| < \epsilon$ and

$$|f(z) > |f(z_0)|, |f(s)| < |f(s_0)|$$

(c) Let f be analytic and nonzero in a region A. Show that $|f|$ has no local minima in A. If f has zeros in A, show that by example that this conlusion does not hold.

[3,6,6]

B8. (a) State Rouche's Theorem.

(b) Show that the equation

$$2z^5 + 8z - 1 = 0$$

has exactly four roots in $1 < |z| < 2$.

(c) Show that if f is analytic inside and on the unit cirle $|z| = 1$, and $0 < |f(z)| < 1$ when $|z| = 1$, then f has exactly one fixed point inside the unit circle.

(d) Show that the equation

$$4z^3 - 12z^2 + 2z + 10 = 0$$

has two roots inside the annulus $1/2 < |z-1| < 2$

[2,4,5,4]

B9. (a) Establish the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

for an analytic function $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$ [3]

(b) Prove that $f(z) = \frac{1}{z-2}$ is analytic in any region not including $z = 2$. [5]

(c) If $v(x,y) = x^2 - y^2 - 2xy - 2x + 3y$,

(i) Show that $v(x,y)$ is harmonic, [2]

(ii) Hence find $u(x,y)$ [3]

(iii) Express $f(x,y) = u(x,y) + iv(x,y)$ as $f(z)$. [2]

B10. Locate the singularities of the function

$$f(z) = \frac{z^6 + 1}{z^3(2z-1)(z-2)}$$

and find the residue at each singularity. Hence show that

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta = \frac{\pi}{12}$$

[15]

B11. If $f(z)$ has a pole of order n at $z = a$, prove that

$$\text{Res} f(z)|_{z=a} = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]$$

If $f(z) = \frac{z^2}{(z-2)(z^2+1)}$ find the residue of $f(z)$ at $z = 2, i, -i$. Determine the Laurent expansion for $f(z)$ about the point $z = 2$, starting where each expansion is valid. Evaluate $\oint_c f(z)dz$ where C is the circle $|z+3| = 2$ described anticlockwise. [15]

B12. Show that the general linear fractional transformation

$$w = \frac{az+b}{cz+d}$$

may be written in terms of just three arbitrary constatnts and deduce that the general formula in terms of three points (z_1, z_2, z_3) and their images (w_1, w_2, w_3) is

$$\frac{(w-w_1)(w_3-w_1)}{(w-w_2)(w_3-w_1)} = \frac{(z-z_1)(z_3-z_1)}{(z-z_2)(z_3-z_1)}.$$

Find the linear fractional transformation which maps $(z_1, z_2, z_3) = (-1, 1, 2)$ onto $(w_1, w_2, w_3) = (0, -1, -3)$.

Find also, for this mapping, the image of the right-hand half $(x > 0)$ of the interior of the unit circle $|z| = 1$ Draw a rough sketch to illustrate your answer. [15]

END OF QUESTION PAPER