

NATIONAL UNIVERSITY OF SCIENCE AND
TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS
August 2004 SUPPLEMENTARY EXAMINATION
SMA2201 COMPLEX ANALYSIS

3 Hours

Answer ALL questions in Section A and any FOUR questions in Section B.

SECTION A : Answer ALL questions from this section.

[28 Marks]

1. Determine if $w(x, y) = x^2 - y^2 + 2x + i(2xy - 2y)$ is analytic.

[3 Marks]

2. Find the residues of the poles of

$$f(z) = \frac{z}{(z-1)(z^2-1)}.$$

[6 Marks]

3. Use the *ML* theorem to bound the integral

$$\oint_{|z|=2} \frac{1}{z^2(z-i)} dz.$$

[4 Marks]

4. Expand $\frac{(1+z)}{(1-z)}$ in a Taylor series about $z=0$ up to the term in z^4 .

[5 Marks]

5. Evaluate $\oint_{|z|=3} \frac{3z}{z^2+2iz-1} dz$.

[4 Marks]

6. Evaluate $\oint_{|z-1|=3} \frac{3z}{(z+4)(z^2+1)} dz$.

[6 Marks]

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SECTION B - Answer any FOUR questions from this section.

[72 Marks]

7. (a) Show that a necessary condition for $w = u(x, y) + iv(x, y)$ to be analytic in a region D is that the Cauchy- Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

be satisfied in the region.

[6 Marks]

- (b) Show that $u = e^{2x} \cos 2y$ is harmonic, and find the simplest conjugate harmonic function v . Express $w = u(x, y) + iv(x, y)$ as $f(z)$.

[6 Marks]

- (c) Find the arcs in the (u, v) plane corresponding to the following arcs in the (x, y) plane.

- i. The straight line $x = 0$.
- ii. The straight line $y = \frac{\pi}{8}$.

[6 Marks]

8. (a) Using Green's Theorem, $\oint_C (Pdx + Qdy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$, show that, if $f(z)$ is analytic within and on a closed contour C , then

$$\oint_C f(z) dz = 0.$$

[6 Marks]

- (b) Show that, for integer n ,

$$\oint_C \frac{1}{(z-a)^n} dz = \begin{cases} 2\pi i & , \quad n = 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

where C is the circle, radius R , centre a .

[6 Marks]

- (c) Hence evaluate $\oint_{|z-1|=3} \frac{1}{(z^2-1)z^2} dz$

[6 Marks]

9. Cauchy's Integral Theorem states that, if $f(z)$ is analytic everywhere within and on a simple closed contour C , then for any point a within C ,

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz.$$

- (a) Show that, for the case $n = 1$,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz.$$

[6 Marks]

- (b) Hence evaluate $\oint_{|z+1|=1} \frac{e^{z/2}}{z^2(2z+3)^2} dz$

[6 Marks]

- (c) Verify Cauchy's Integral Theorem for $f(z) = z$, $a = 1$ and C the path around the circle, centre 1, radius 1.

[6 Marks]

10. (a) Derive the Laurent Series formula for a function $f(z)$ that is single-valued and analytic in the annulus $r < z - a < R$,

$$f(z) = \sum_{n=1}^{\infty} a_n(z-a)^n,$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \quad (n = 0, \pm 1, \pm 2, \dots),$$

where C is a circle, centre a , lying completely inside the annulus.

[6 Marks]

- (b) Hence find the Laurent Series for

i. $\frac{\cos 4z}{z^2}$ about $z = 0$,

[6 Marks]

ii. $\frac{1}{(z^2+1)^2}$ for $0 < |z-i| < 2$.

[6 Marks]

11. (a) Show that, if f has a pole of order n at $z = a$, then

$$\text{Res } f(a) = \frac{1}{(n-1)!} \lim_{z \rightarrow a} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z).$$

[6 Marks]

- (b) Hence evaluate

i. $\int_0^{\infty} \frac{1}{(x^2+4)^2} dx$

[6 Marks]

ii. $\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 2x + 2} dx.$

[6 Marks]

12. (a) Define the Cauchy Principle Value of $\int_C f(z) dz$ and hence show that, if R is the residue of $f(z)$ at the point a , which is a simple pole on the contour C , then

$$P \oint_C f(z) dz = \pi i R.$$

[6 Marks]

(b) Hence evaluate $\int_{-\infty}^{\infty} \frac{1}{(x^2 + 2x - 3)} dx.$

[4 Marks]

- (c) If $f(z)$ is analytic in a closed domain D , except at a finite number of poles, and $f(z)$ has neither zeros nor poles on the boundary C of D , then

$$\oint_C \frac{f'(z)}{f(z)} dz = 2\pi i(N - P),$$

where N and P are the number of zeros and poles of $f(z)$ respectively in D .

Hence evaluate $\oint_{|z|=7} \frac{\sec^2 z}{\tan z} dz.$

[8 Marks]

END OF EXAMINATION PAPER

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