NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

May 2005 EXAMINATION

SMA2201 COMPLEX ANALYSIS

3 Hours

Answer ALL questions in Section A and any FOUR questions in Section B.

SECTION A : Answer ALL questions from this section.

[28 Marks]

1. Find the order of the poles of $f(z) = \dfrac{z}{(\sin z)^2}$.

[4 Marks]

2. Find the residues of the poles of

$$f(z) = \frac{\cos \pi z}{(z+1)(z^2-1)}.$$

[6 Marks]

3. Use the ML theorem to bound the integral

$$\oint_{|z|=2} \frac{z+i}{z(z-i)}\, dz.$$

[4 Marks]

4. Expand $\dfrac{e^{iz}}{z^2}$ in a Laurent series about $z=0$ up to the term in z^2.

[4 Marks]

5. Evaluate $\displaystyle\oint_{|z-1-i|=2} \frac{e^z}{(z^2+1)^2}\, dz$.

[5 Marks]

6. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx$.

[5 Marks]

SECTION B - Answer any FOUR questions from this section.

[72 Marks]

7. (a) Show that a necessary condition for $w = u(x,y) + iv(x,y)$ to be analytic in a region D is that the Cauchy- Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

be satisfied in the region.

[6 Marks]

(b) Show that $u = e^{-y}(x\cos x - y\sin x)$ is harmonic, and find a conjugate harmonic function v. Express $w = u(x,y) + iv(x,y)$ as $f(z)$, given that $f(i) = ie^{-1}$.

[8 Marks]

(c) Show that any function $w = f(z^*)$, where z^* is the complex conjugate of z, does not satisfy the Cauchy- Riemann equations.

[4 Marks]

8. (a) Using Green's Theorem, $\oint_C (Pdx + Qdy) = \int\int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\, dxdy$, show that, if $f(z)$ is analytic within and on a closed contour C, then

$$\oint_C f(z)\, dz = 0.$$

[6 Marks]

(b) Show that, for integer n,

$$\oint_C \frac{1}{(z-a)^n}\, dz = \begin{cases} 2\pi i \ , & n = 1 \\ 0 \ , & otherwise \end{cases}$$

where C is the circle, radius R, centre a.

[6 Marks]

(c) Hence use partial fractions to evaluate

$$\oint_{|z+1|=2} \frac{z+1}{(z-2)z^3}\, dz.$$

[6 Marks]

9. Cauchy's Integral Theorem states that, if $f(z)$ is analytic everywhere within and on a simple closed contour C, then for any point a within C,

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz.$$

(a) Use Cauchy's Integral Theorem to derive Cauchy's Extended Integral Theorem,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz,$$

for the case $n = 1$. [6 Marks]

(b) Hence evaluate $\displaystyle\oint_{|z+1|=2} \frac{e^{(z/2)}}{(z)(z+2)^2}\, dz.$ [6 Marks]

(c) Verify Cauchy's Integral Theorem for $f(z) = z + 1$, $a = 1$ and C the path around the circle, centre 1, radius 1. [6 Marks]

10. (a) Derive the Laurent Series formula for a function $f(z)$ that is single-valued and analytic in the annulus $r < |z - a| < R$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n,$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz \quad (n = 0, \pm 1, \pm 2, \ldots),$$

where C is a circle, centre a, lying completely inside the annulus. [6 Marks]

(b) Hence find the Laurent Series for

i. $\dfrac{\sinh z}{z^3}$ about $z = 0$, [6 Marks]

ii. $\dfrac{1}{z(z+1)^2}$ about $z = -1$. [6 Marks]

11. (a) Show that, if f has a pole of order n at $z = a$, then

$$Res\ f(a) = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z).$$

[6 Marks]

(b) Hence evaluate

i. $\displaystyle\int_0^{\infty} \frac{1}{(x^2+1)^2}\, dx$

[6 Marks]

ii. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2 - 2x + 5}\, dx.$

[6 Marks]

12. (a) Define the Cauchy Principle Value of $\int_C f(z)\, dz$ and hence show that, if R is the residue of $f(z)$ at the point a, which is a simple pole on the contour C, then

$$P \oint_C f(z)\, dz = \pi i R.$$

[7 Marks]

(b) Hence evaluate

i. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+2x-3)}\, dx$

[5 Marks]

ii. $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{(x-1)(x^2+1)}\, dx$

[6 Marks]

END OF EXAMINATION PAPER