NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

2005 SUPPLEMENTARY EXAMINATION

SMA2201 COMPLEX ANALYSIS

3 Hours

Answer ALL questions in Section A and any FOUR questions in Section B.

SECTION A : Answer ALL questions from this section.

[28 Marks]

1. Determine if $w(x, y) = x^2 - y^2 - 2x + i(2xy - 2y)$ is analytic.

[3 Marks]

2. Find the residues of the poles of

$$f(z) = \frac{z^4 - 1}{(z+1)^2(z^2+1)}.$$

[6 Marks]

3. Use the ML theorem to bound the integral

$$\oint_{|z|=6} \frac{1}{z(z-5)}\, dz.$$

[4 Marks]

4. Expand $\sin \frac{z}{(1-z)}$ in a Taylor series about $z = 0$ up to the term in z^5.

[5 Marks]

5. Evaluate $\oint_{|z|=3} \frac{3}{z(z^2 - 6z + 5)}\, dz.$

[4 Marks]

6. Evaluate $\oint_{|z-2|=4} \frac{3z+2}{(z+4)(z+1)^2(z^2+1)}\, dz.$

[6 Marks]

SECTION B - Answer any FOUR questions from this section.

[72 Marks]

7. (a) Show that a necessary condition for $w = u(x,y) + iv(x,y)$ to be analytic in a region D is that the Cauchy- Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

be satisfied in the region.

[6 Marks]

(b) Show that $u = 2xy$ is harmonic, and find the simplest conjugate harmonic function v. Express $w = u(x,y) + iv(x,y)$ as $f(z)$.

[6 Marks]

(c) Find the arcs in the (u, v) plane corresponding to the following arcs in the (x, y) plane.

i. The circle $x = \cos t, \ y = \sin t, \ \ 0 < t \leq 2\pi$
ii. The straight line $y = kx$.

[6 Marks]

8. (a) Using Green's Theorem, $\oint_C (Pdx + Qdy) = \int\int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\, dxdy$, show that, if $f(z)$ is analytic within and on a closed contour C, then

$$\oint_C f(z)\, dz = 0.$$

[6 Marks]

(b) Show that, for integer n,

$$\oint_C \frac{1}{(z-a)^n}\, dz = \begin{cases} 2\pi i \ , & n = 1 \\ 0 \ , & otherwise \end{cases}$$

where C is the circle, radius R, centre a.

[6 Marks]

(c) Hence evaluate $\oint_{|z-1|} = 3 \frac{1}{(z^2-1)(z+1)(z^3)}\, dz.$

[6 Marks]

9. Cauchy's Integral Theorem states that, if $f(z)$ is analytic everywhere within and on a simple closed contour C, then for any point a within C,

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz.$$

(a) Show that, for the case $n = 1$,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz.$$

[6 Marks]

(b) Hence evaluate $\oint_{|z+1|=1} \frac{\sin(\pi z/2)}{(z^2-1)(2z+3)^2}\, dz.$

[6 Marks]

(c) Verify Cauchy's Integral Theorem for $f(z) = 2z + 1$, $a = 1$ and C the path around the circle, centre 1, radius 2.

[6 Marks]

10. The Laurent Series formula for a function $f(z)$ that is single-valued and analytic in the annulus $r < z - a < R$ is given by

$$f(z) = \sum_{n=1}^{\infty} a_n (z-a)^n,$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz \quad (n = 0, \pm 1, \pm 2, \ldots),$$

where C is a circle, centre a, lying completely inside the annulus.

[6 Marks]

Hence find the first 5 terms in the Laurent Series for

(a) $\dfrac{\cos 3z}{z^2}$ about $z = 0$,

[6 Marks]

(b) $\dfrac{1}{(z^2-1)}$ for $0 < z - 1 < 2$.

[6 Marks]

(c) $\cosh z$ for $z < \pi$.

[6 Marks]

11. (a) Show that, if f has a pole of order n at $z = a$, then

$$Res\ f(a) = \frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}(z-a)^n f(z).$$

[6 Marks]

(b) Hence evaluate

i. $\displaystyle\int_0^{\infty}\frac{\cos x}{(x^2+4)^2}\,dx$

[6 Marks]

ii. $\displaystyle\int_{-\infty}^{\infty}\frac{x\sin 5x}{x^2-2x+5}\,dx.$

[6 Marks]

12. (a) Define the Cauchy Principle Value of $\int_C f(z)\,dz$ and hence show that, if R is the residue of $f(z)$ at the point a, which is a simple pole on the contour C, then

$$P\oint_C f(z)\,dz = \pi i R.$$

[6 Marks]

(b) Hence evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{x+2}{x(x^2-1)}\,dx.$

[4 Marks]

(c) If $f(z)$ is analytic in a closed domain D, except at a finite number of poles, and $f(z)$ has neither zeros nor poles on the boundary C of D, then

$$\oint_C \frac{f'(z)}{f(z)}\,dz = 2\pi i(N-P),$$

where N and P are the number of zeros and poles respectively in D.
Hence evaluate $\displaystyle\oint_{|z|=10}\tanh z\,dz.$

[8 Marks]

END OF EXAMINATION PAPER