NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 2206

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

NUMERICAL ANALYSIS

MAY 2002

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A: Answer ALL questions in this section [40].

A1. For the equation

$$x = \frac{2 - e^x + x^2}{3},$$

determine an interval $[a, b]$ for which the fixed point iteration will converge. [5]

A2. Apply the LU-decomposition method to solve the equations

$$3x + 2y + 7z = 4, \quad 2x + 3y + z = 5, \quad 3x + 4y + z = 7.$$

[7]

A3. Consider the initial value problem

$$y' = 1 + xy^2, \quad y(0) = 1.$$

Use the Euler-Cauchy method to compute the approximation to $y(0.2)$ taking $h = 0.1$. [6]

A4. The voltage $E = E(t)$ in an electric circuit obeys the equation

$$E(t) = L\frac{dI}{dt} + RI(t).$$

where R is the resistance and L is the inductance. Use $L = 0.98$ and $R = 0.142$ and the values of $I(t)$ in the table below

t	1.00	1.01	1.02	1.03	1.04
$I(t)$	3.10	3.12	3.14	3.18	3.24

to compute $E(1.02)$. [5]

A5. Simpson's rule with the error term takes the form

$$\int_{x_0}^{x_2} f(x)dx = a_0 f_0 + a_1 f_1 + a_2 f_2 + k f^{(IV)}(\xi).$$

Find a_0, a_1 and a_2 from the fact that Simpson's rule is exact for $f(x) = x^n$ when $n = 1, 2, 3$. Then find k by applying the integration formula with $f(x) = x^4$. [6]

A6. Use the Gauss-Seidel iteration method to approximate the solution of the following system correct to 2 decimal places with $x_0 = y_0 = z_0 = 0$ as initial values.

$$2x + 16y + z = 39, \quad x + y + 25z = 83, \quad 10x + y + z = 19.$$

[6]

A7. Use Lagrange interpolating formula inversely to obtain a root of the equation $f(x) = 0$ to 3 decimal places given that

$$f(-2) = -31, \quad f(-1) = -5, \quad f(1) = 1, \quad f(2) = 11.$$

[5]

SECTION B: Answer THREE questions in this section [60].

B8. (a) Prove that if $f \in [a, b]$ and $f(a) \cdot f(b) < 0$, then the Bisection method generates a sequence $\{x_n\}$ approximating the exact solution x, with the property that

$$|x_n - x| \leq \frac{b-a}{2^n}, \quad n \geq 1.$$

[5]

(b) A trough of length L has a cross-section in the shape of a semi-circle of radius r. When filled with water to within a distance h of the top, the volume V of water

$$V = L[0.5\pi r^2 - r^2 \arcsin(h/r) - h(r^2 - h^2)^{\frac{1}{2}}].$$

Suppose $L = 10$ feet, $r = 1$ foot and $V = 12.4$ cubic feet. Use the Bisection method to find the depth of water in the trough to within 0.01 feet. [7]

(c) Let $g \in C[a,b]$ and suppose that $g(x) \in [a,b]\ \forall\ x \in [a,b]$. Further, suppose g' exists on (a,b) with

$$|g'(x)| \leq k < 1 \ \ \forall x \in (a,b).$$

If x_0 is any number in $[a,b]$, then the sequence defined by

$$x_n = g(x_{n-1}), \quad n \geq 1$$

converges to the unique fixed point x in $[a,b]$. Prove this theorem. [8]

B9. (a) For a system $A\mathbf{x} = \mathbf{b}$, the residual is given by $\mathbf{r} = A\mathbf{e}$ where $\mathbf{e}$ is the error in the computed solution $\hat{\mathbf{x}}$. Show that the error bounds of the relative error $\frac{||\mathbf{e}||}{||\mathbf{x}||}$ in terms of the relative residual $\frac{||\mathbf{r}||}{||\mathbf{b}||}$ and condition number $\kappa(A)$ are

$$\frac{1}{\kappa(A)} \cdot \frac{||\mathbf{r}||}{||\mathbf{b}||} \leq \frac{||\mathbf{e}||}{||\mathbf{x}||} \leq \kappa(A) \cdot \frac{||\mathbf{r}||}{||\mathbf{b}||}$$

given that $\kappa(A) \geq 1$. [6]

(b) Use the following values to construct a cubic Lagrange polynomial approximation to $f(1.09)$.

x	1.00	1.05	1.10	1.15
$f(x)$	0.1924	0.2414	0.2933	0.3492

The function being approximated is $f(x) = \log_{10} \tan x$. Use this knowledge to find a bound for the error in the approximation. [7]

(c) Let $D_0(h)$ denote the approximation to $f'(x)$ obtained from the three-point central difference formula with step size h:

$$D_0(h) = \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

and thus $f'(x_0)$ can be expressed as

$$f'(x_0) = D_0(h) + Ch^2$$

where C is some constant. Derive a five-point central difference formula by the Richardson extrapolation method using a step size of $2h$. [7]

B10. (a) Given the initial value problem

$$y' = \frac{2}{x}y + x^2e^x, \quad 1 \le x \le 2, \quad y(1) = 0$$

with exact solution $y(x) = x^2(e^x - e)$, use Taylor method of order two with $h = 0.1$ to approximate the solution and compare it with actual values of y. [10]

(b) Apply 3 steps of the Runge-Kutta method of fourth order with $h = 0.2$ to the following initial value problem.

$$y' = (1 + x^{-1})y, \quad y(1) = e.$$

Solve the problem analytically and compute errors. [10]

B11. (a) Find an approximation to the area of the region bounded by the normal curve

$$y = \frac{1}{\sigma\sqrt{2\pi}}e^{(x/\sigma)^2/2}$$

and the x-axis on the interval $[-\sigma, \sigma]$ using the Composite Trapezoidal rule with $n = 8$. [8]

(b) The Gaussian quadrature formula for $N = 3$ is given by

$$\int_{-1}^{1} f(x)dx = \alpha_1 f(x_1) + \alpha_2 f(x_2) + \alpha_3 f(x_3) + \text{Error}$$

Find the coefficients α_1, α_2 and α_3 and the quadrature points x_1, x_2 and x_3 so that this method is exact for polynomials of as high a degree as possible.
Use this method to evaluate the integral

$$I = \int_0^1 e^{-x^2}dx.$$

[12]

END OF QUESTION PAPER