NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 2206

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

NUMERICAL ANALYSIS

JUNE 2004

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A: Answer ALL questions in this section [40].

A1. Consider the matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

(a) Show that A cannot be factored into the product of a unit lower triangular matrix and an upper triangular matrix.

(b) Interchange the rows of A so that this can be done. [4,2]

A2. The function $f(x) = e^x - 2 - x$ has one real root in the interval $[1.0, 2.8]$. How many bisections would be required to locate this root to an error $\epsilon = 0.5 \times 10^{-5}$? [5]

A3. Derive the numerical differentiation approximation formula

$$f'(x) \approx \frac{1}{2h}[4f(x+h) - 3f(x) - f(x+2h)]$$

and determine its error term. [7]

A4. Find an approximation to the area of the region bounded by the normal curve

$$y = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{(x/\sigma)^2/2}$$

and the x-axis on the interval $[-\sigma, \sigma]$ using the Trapezoidal rule with $n = 8$. [8]

A5. Construct Newton's interpolating polynomial for the data

x	0	2	3	4
y	7	11	28	63

[7]

A6. Solve the following system by Gaussian elimination method with scaled partial pivoting

$$\begin{bmatrix} 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \\ 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -19 \\ -34 \\ 16 \\ 26 \end{bmatrix}$$

[7]

SECTION B: Answer FOUR questions in this section [60].

B7. (a) Given the initial value problem

$$y' = \frac{2}{x}y + x^2 e^x, \; x \in [1, 2], \; y(1) = 0$$

with exact solution $y(x) = x^2(e^x - e)$, use Taylor's method of order two with $h = 0.2$ to approximate the solution and compare it with actual values of y. [10]

(b) Apply three steps of the Runge-Kutta method of fourth order with $h = 0.2$ to the following initial value problem:

$$y' = (1 + x^{-1})y, \; y(1) = e.$$

Solve the problem analytically and compute the errors. [10]

B8. (a) (i) Verify that when Newton-Raphson method is used to compute $\sqrt{R}$, the sequence of iterates is defined by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right).$$

(ii) Show that if the sequence $\{x_n\}$ is defined as above, then

$$x_{n+1}^2 - R = \left(\frac{x_n^2 - R}{2x_n}\right)^2.$$

[5,5]

(b) Use the bisection method to find the root of

$$\ln\left(\frac{1+x}{1+x^2}\right) = 0$$

in the interval $(0.99, 1.01)$. What is the other root? [5]

(c) Calculate an approximate value for $4^{3/4}$ using three steps of the secant method with $x_0 = 3$ and $x_1 = 2$. [5]

B9. (a) Using five points, apply Simpson's rule to approximate π from the formula

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}.$$

[5]

(b) Construct a numerical integration rule of the form

$$\int_{-1}^{1} f(x)dx \approx \alpha f(-1/2) + \beta f(0) + \gamma f(1/2)$$

that is axact for all polynomials of degree equal to or less than 2. [7]

(c) Determine the quadrature formula of the form

$$\int_a^b f(x)dx \approx w_0 f(a) + w_1 f(b) + w_2 f'(a) + w_3 f'(b)$$

that is exact for polynomials of as high a degree as possible. [8]

B10. (a) A trough of length L has a cross-section in the shape of a semi-circle of radius r. When filled with water to within a distance h of the top, the volume V of the water is given by

$$V = L[0.5\pi r^2 - r^2 \arcsin(h/r) - h(r^2 - h^2)^{\frac{1}{2}}].$$

Suppose $L = 10$ metres, $r = 1$ foot and $V = 12.4$ cubic metres. Use the bisection method to find the depth of water in the trough to within 0.01 metres. [7]

(b) For a system $A\mathbf{x} = \mathbf{b}$, the residual is given by $\mathbf{r} = A\mathbf{e}$ where $\mathbf{e}$ is the error in the computed solution $\tilde{\mathbf{x}}$. Show that the error bounds of the relative error, $\frac{||\mathbf{e}||}{||\mathbf{x}||}$ in terms of the relative residual $\frac{||\mathbf{r}||}{||\mathbf{b}||}$ and the condition number $\kappa(A)$ are

$$\frac{1}{\kappa(A)} \cdot \frac{||\mathbf{r}||}{||\mathbf{b}||} \le \frac{||\mathbf{e}||}{||\mathbf{x}||} \le \kappa(A) \cdot \frac{||\mathbf{r}||}{||\mathbf{b}||}$$

given that $\kappa(A) \ge 1$.

[6]

(c) Use the following values to construct a cubic Lagrange polynomial approximation to $f(1.09)$.

x	1.00	1.05	1.10	1.15
$f(x)$	0.1924	0.2414	0.2933	0.3492

The function being approximated is $f(x) = \log_{10} \tan x$. Use this knowledge to find a bound for the error in the approximation. [7]

END OF QUESTION PAPER

DATA SHEET

1. **Runge-Kutta Method of Order Four:**

$$y_{k+1} = y_k + \frac{1}{6}h[k_1 + 2k_2 + 2k_3 + k_4]$$

where:

$$\begin{aligned} k_1 &= f(x_k, y_k) \\ k_2 &= f(x_k + \frac{1}{2}h, y_k + \frac{1}{2}hk_1) \\ k_3 &= f(x_k + \frac{1}{2}h, y_k + \frac{1}{2}hk_2) \\ k_4 &= f(x_k + h, y_k + hk_3) \end{aligned}$$

2. **Second Order Taylor Series Method:**

$$y_{k+1} = y_k + hf(x_k, y_k) + \frac{1}{2}h^2[f_x(x_k, y_k) + f_y(x_k, y_k)f(x_k, y_k)]$$