NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 2206

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

NUMERICAL ANALYSIS

MAY 2005

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A: Answer ALL questions in this section [40].

A1. Use the following values to construct a cubic Lagrange polynomial approximation to $f(1.09)$.

x	1.00	1.05	1.10	1.15
$f(x)$	0.1924	0.2414	0.2933	0.3492

The function being approximated is $f(x) = \log_{10} \tan x$. Use this knowledge to find a bound for the error in the approximation. [7]

A2. Find an approximation to the area of the region bounded by the normal curve

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{1}{2}(\frac{x}{\sigma})^2}$$

and the $x-$axis on the interval $[-\sigma, \sigma]$ using the trapezoidal rule with n=8. [8]

A3. Determine the quadrature formula of the form

$$\int_a^b f(x)dx \approx w_0 f(a) + w_1 f(b) + w_2 f'(a) + w_3 f'(b).$$

that is exact for polynomials of as high a degree as possible. [8]

A4. Consider the matrix:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

(a) Show that A cannot be factored into the product of a unit lower triangular matrix and an upper triangular matrix.

(b) Interchange the rows of A so that this can be done. [4,2]

A5. The function $f(x) = e^x - 2 - x$ has one real root in the interval $[1.0, 2.8]$. How many bisections would be required to locate this root to an error $\epsilon = 0.5 \times 10^{-5}$? [5]

A6. For a system $A\mathbf{x} = \mathbf{b}$, the residual is given by $\mathbf{r} = A\mathbf{e}$ where $\mathbf{e}$ is the error in the computed solution $\tilde{\mathbf{x}}$. Show that the error bounds of the relative error, $\frac{||\mathbf{e}||}{||\mathbf{x}||}$ in terms of the relative residual $\frac{||\mathbf{r}||}{||\mathbf{b}||}$ and the condition number $\kappa(A)$ are:

$$\frac{1}{\kappa(A)} \cdot \frac{||\mathbf{r}||}{||\mathbf{b}||} \le \frac{||\mathbf{e}||}{||\mathbf{x}||} \le \kappa(A) \cdot \frac{||\mathbf{r}||}{||\mathbf{b}||}$$

given that $\kappa(A) \ge 1$. [6]

SECTION B: Answer THREE questions in this section [60].

B7. (a) Apply Gaussian Elimination method with scaled partial pivoting to solve the system:

$$\begin{aligned} 2x + y + z + 3w &= 7 \\ x + 4y + 3z + w &= 9 \\ 2x - y - 5z + w &= -3 \\ 3x - 4y + 2z + 8w &= 9 \end{aligned}$$

[6]

(b) Prove that

$$||\mathbf{x}^{(k)} - \mathbf{x}|| \le ||B||^k ||\mathbf{x}^{(0)} - \mathbf{x}||$$

where $B \in \Re^{n \times n}$ with $||B|| < 1$ and

$$\mathbf{x}^{(k)} = B\mathbf{x}^{(k-1)} + \mathbf{c}, \quad k = 1,\ 2, \cdots$$

with $\mathbf{x}^{(0)}$ arbitrary, $\mathbf{c} \in \Re^n$, and $\mathbf{x} = B\mathbf{x} + \mathbf{c}$. [6]

(c) Consider the set of equations:

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Suppose the true solutions of these equations x, y and z and the iterative values after r steps of the Jacobi process are x_r, y_r and z_r where

$$x_r = x + \alpha_r \quad y_r = y + \beta_r \quad z_r = z + \gamma_r.$$

Let $E_r = \max\{|\alpha_r|, |\beta_r|, |\gamma_r|\}$, i.e. the largest error after r steps.

(i) Prove that

$$|\alpha_{r+1}| \leq \left|\frac{a_{12}}{a_{11}}\right| |\beta_r| + \left|\frac{a_{13}}{a_{11}}\right| |\gamma_r|$$
$$|\beta_{r+1}| \leq \left|\frac{a_{21}}{a_{22}}\right| |\alpha_r| + \left|\frac{a_{23}}{a_{22}}\right| |\gamma_r|$$
$$|\gamma_{r+1}| \leq \left|\frac{a_{31}}{a_{33}}\right| |\alpha_r| + \left|\frac{a_{32}}{a_{33}}\right| |\gamma_r|$$

(ii) Comment on the convergence scheme if

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

[8]

B8. (a) Solve the following initial value problem

$$y'(x) = 1 + xy^2, \quad y(0) = 1$$

(i) by the Euler-Cauchy method.

(ii) by the Fourth Order Runge-Kutta method.

using $h = 0.1$ to estimate $x(0.2)$. Compare the results. [6,7]

(b) Derive the following rule for estimating the third derivative

$$f'''(x) \approx \frac{1}{2h^3}[f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)]$$

and determine the error term. [7]

B9. (a) (i) Verify that when Newton-Raphson method is used to compute $\sqrt{R}$, the sequence of iterates is defined by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right).$$

(ii) Show that if the sequence $\{x_n\}$ is defined as above, then

$$x_{n+1}^2 - R = \left(\frac{x_n^2 - R}{2x_n}\right)^2.$$

[5,5]

(b) Use the bisection method to find the root of

$$\ln\left(\frac{1+x}{1+x^2}\right) = 0$$

in the interval $(0.99, 1.01)$. What is the other root? [5]

(c) Calculate an approximate value for $4^{3/4}$ using three steps of the secant method with $x_0 = 3$ and $x_1 = 2$. [5]

B10. (a) Use the divided difference method to obtain a polynomial of least degree that fits the values given in the following table:

x	0	1	2	-1	3
f(x)	-1	-1	-1	-7	5

[7]

(b) Using five points, apply Simpson's rule to approximate π from the formula

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}.$$

[6]

(c) Construct a numerical integration rule of the form

$$\int_{-1}^{1} f(x)dx \approx \alpha f(-1/2) + \beta f(0) + \gamma f(1/2)$$

that is axact for all polynomials of degree equal to or less than 2. [7]

END OF QUESTION PAPER