NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

SUPPLEMENTARY EXAMINATION July 2001

SMA2209 TOPICS IN APPLIED MATHEMATICS

3 Hours

This paper contains TWO sections. Answer ALL the questions in section A and THREE questions from section B.

SECTION A : **Answer ALL questions from this section.**

1. The velocity of a object falling from rest through air is thought to depend principally on the acceleration due to gravity and the distance fallen. Use dimensional analysis to find an expression for the velocity in terms of the other factors.

[4 Marks]

2. Write the differential equation for the forced damped motion of a pendulum

$$m\frac{d^2y}{dt^2} + 2r\frac{dy}{dt} + \omega^2 y = F(t)$$

in dimensionless form.

[4 Marks]

3. Write down a constant, a parameter, an input variable and an output variable that you think may be significant in modelling the flow of a viscous liquid under gravity through a pipe.

[4 Marks]

4. Given that 0.456 and 0.696 are random numbers from a uniform distribution on (0,1),

   (a) generate two random numbers from a uniform distribution on (3,7),

   [2 Marks]

   (b) generate two random numbers from a Poisson distribution of mean 0.5.

   [4 Marks]

5. Write down mathematical expressions to model the following situations.

   (a) The temperature rose from zero to reach a maximum of $20°C$ after 5 minutes, before decreasing eventually to zero.

   [3 Marks]

   (b) The temperature increased from an initial value of $20°C$ to eventually reach $100°C$.

   [3 Marks]

   (c) The temperature varied sinusoidally between $10°C$ and $20°C$ with a period of one day, being $10°C$ initially.

   [4 Marks]

SECTION B : Answer THREE questions from this section.

6. The water supply for a particular town is provided by a dam situated in a gorge near to the town. The town council would like an estimate of how the depth of water in the dam is likely to change throughout the dry season, during which time no significant rainfall is expected. The council expects to take water along the pipeline from the dam at an approximately constant rate throughout the dry season. In cross section, the floor of the dam rises symmetrically from the centre line at an angle ϕ. Moving away from the dam wall, the centre line of the floor slopes upwards at an approximately constant angle θ.

   (a) Show that the volume of water in the dam at a given time is $k_1 h^3$, where h is the maximum depth of water in the dam at that time, and find the parameter k_1 in terms of the width of ϕ and θ.

   [4 Marks]

(b) Assuming that the only loss of water from the dam is through the pipeline, use conservation of fluid volume to show that a simple model for the rate of change of depth of water in such a dam is

$$\frac{dh}{dt} = -\frac{k_2}{h^2},$$

where k_2 is a parameter.

Hence find an expression for the depth of fluid in the dam in terms of time and the parameters of the dam and water supply.

[7 Marks]

(c) In one particular dry season, the maximum depth of water at the dam wall as a function of time was as follows.

Depth(m)	35	30	25	20	15
Time(months)	2.1	4.2	6.0	7.5	8.8

By drawing an appropriate graph, use these data to check the validity of the model developed in (b).

[5 Marks]

(d) An alternative model of the depth of water would include loss of water through evaporation. Stating clearly any assumptions you make, derive an alternative differential equation for the rate of change of depth of water. Solve this differential equation and test your model using the data in (c). Which model would you recommend to the council?

[8 Marks]

7. A medical research centre is investigating the transmission of an infectious disease in small isolated communities. For the rapidly transmitted disease under consideration, the total number of people in the community can be considered constant during a particular episode of the disease, i.e. no births, deaths, immigration or emigration are likely to take place during the episode. If no treatment is given, anyone with the disease remains infectious and can pass the disease to anyone in the community who has not been exposed to the disease. Past studies have shown that the rate at which the number of infectious people in a community increases is proportional to the product of the number of infectious people and the number of people who do not have the disease but are liable to infection.

(a) Stating clearly any assumptions made and defining variables and parameters carefully, show that the rate of change of the number of infectious people in the community could be written as

$$\frac{dI}{dt} = k_1 I(N - I),$$

where k_1 and N are parameters.

[2 Marks]

(b) Solve the equation in (a) to find $I(t)$, given that one person is initially infected in a community of 1000 people. The medical centre estimates that, if I is very small compared to N , each infected person passes on the infection at a rate of one person every 2 weeks.

[6 Marks]

(c) Several treatments are available for people with the disease. One treatment involves giving a drug to all infected persons. A fixed percentage of the infected people will then be cured, and cease to be infectious, each week. However, those cured are not immune from being infected by the disease again. Indeed they seem to be as susceptible as those who have never been exposed to the disease. Show that, if this treatment was given from the beginning, the rate of change of the number of infectious people in the community could be written as

$$\frac{dI}{dt} = k_1 I(1 - I/N) - k_3 I,$$

where k_3 is a parameter. Solve this differential equation and comment on the nature of the solution for different values of the parameters and initial conditions.

[8 Marks]

(d) An alternative treatment will similarly cure a fixed percentage of those infected each week. These will then cease to be infectious and will also be immune from further infection. Deduce a pair of differential equations to model the spread of the disease in a community undergoing this treatment from the beginning. Comment on the nature of the solution for different values of the parameters and initial conditions. Describe briefly how you would use a spreadsheet to investigate solutions of the equations.

[8 Marks]

8. A local post office deals with two types of transactions, which can be classified as A, the issue of postage stamps only, and B, other transactions. The manager of the post office has asked a management consultant to advise her. There are only two counters available at the post office but the manager could organise the counter staff in various ways by restricting one or more of the counters to specialist use: that is, using that counter for only one of A and B. A survey of the office over a period of several days showed that 40 per cent of the customers came for stamps only. The other customers were classified as having other transactions. The time between arrivals of customers, x, closely followed an exponential distribution with a mean of 50 seconds, i.e.

$$p(x) = \frac{e^{-\frac{x}{50}}}{50}.$$

The time to serve customers, y, closely followed a uniform distribution, i.e.

$$\begin{aligned} p(y) &= 0 && ; \; y < a_1 \\ &= \frac{1}{a_2 - a_1} && ; \; a_1 < y < a_2 \\ &= 0 && ; \; y > a_2 \end{aligned}$$

For both types of transactions a_1 is 10 seconds, but a_2 varies with the type of transaction, being 60 seconds for type A transactions and 140 seconds for type B transactions.

The management consultant first simulates the operation of the Post Office when none of the counters are in specialist use. The consultant uses the MINITAB statistical package to generate the following types of transaction, inter-arrival times and service times for the first ten customers to arrive after opening.

Customer	Transaction	Inter-arrival Time	Service Time
1	A	37	33
2	B	13	115
3	B	28	87
4	A	46	38
5	A	22	49
6	B	49	66
7	B	37	88
8	A	2	19
9	B	33	47
10	B	17	28

(a) Write down the MINITAB commands which would produce information similar to that given above.

[4 Marks]

(b) Use the random numbers from a uniform distribution on (0,1) given in the table below to generate similar information for the next five customers to arrive.

0.676	0.347	0.557
0.177	0.971	0.693
0.278	0.079	0.801
0.692	0.576	0.632
0.811	0.662	0.088

[5 Marks]

(c) Stating clearly any assumptions you make, simulate the service of these fifteen customers in the Post Office. Estimate the mean time and the maximum time that these customers spend in the Post Office.

[8 Marks]

(d) An alternative strategy would be to use one of the counters for the issue of postage stamps only, keeping the other counter for the other transactions. Using the same random numbers as before, simulate the service of the fifteen customers under the new strategy. Comment on the effect on customer service of adopting this alternative strategy.

[7 Marks]

9. Devise a mathematical model for **one** of the following problems.

(a) Trucks driving at night through a city cross through several junctions controlled by robots. At these times there are rarely other vehicles on the road. A transport company wishes to advise their drivers on how to adjust their speed after initially seeing a robot so as to minimise wasting fuel by unnecessarily decelerating and accelerating.

[24 Marks]

(b) It has been claimed that, due to the present rapid rise in human population, there are more people alive today than the total number of people who haved died throughout history. Investigate this claim. You may assume that rose from 2 people to the present 6,000,000,000 in 500,000 years with approximately constant birth and death rates.

[24 Marks]

(c) A coin is dropped from a given height onto a soft surface so that no bouncing occurs. It is released horizontally, head upwards, but is flipped to give a constant rate of rotation. Under what initial conditions will it land with head upwards?

[24 Marks]

END OF EXAMINATION PAPER